Foundations of mathematics

أسس الرياضيات

Foundations of mathematics is the study of the basic mathematical concepts (logic statements, numbers, relations, sets functions...)

أسس الرياضيات هو علم دراسة المفاهيم الرياضية الأساسية كالعبارات المنطقية ، أنظمة الأعداد، العلاقات ، المجاميع ، الدوال ...

<u>Set of Numbers (Subsets of the set of real numbers R)</u>

1. Set of Natural numbers $N = \{1,2,3,\ldots\}$.
2. Set of Prime numbers $P = \{2,3,5,7,11,\ldots\}$.
3. Set of Integer numbers $Z = I = \{\ldots,-2,-1,0,1,2,\ldots\}$.
4. Set of Even numbers $E = \{\ldots,-4,-2,0,2,4,\ldots\}$.
5. Set of Odd numbers $O = \{\ldots,-3,-1,1,3,\ldots\}$.
6. Set of Rational numbers $Q = \{ a/b : a, b \in Z , b \neq 0\}$.
7. Set of Irrational numbers $H = \{x: x \notin Q\}$.

<u>Example:</u>

$2/3, -1/5, 3, 0.5, 0.3333$ are examples of rational numbers.

<u>Example:</u>

$\pi = 3.1415\ldots$. is irrational number.

$e = 2.71828\ldots$ is irrational number.

$\sqrt{2}, \sqrt{5}$ are irrational numbers.

CHAPTER ONE

Mathematical Logic المنطق الرياضي

Contents:

1. Propositions (statements) العبارات

2. Compound propositions العبارات المركبة

3. Mathematical proof البرهان الرياضي

4. Quantifiers المسورات

Definition: Mathematical Logic is a subfield mathematics exploring the applications of formal logic to mathematics. Mathematical logic are widely used in theoretical computer science and other sciences.

المنطق الرياضي هو احد الحقول الرياضية التي تدرس تطبيقات المنطق في الرياضيات. المنطق الرياضي يستخدم بشكل واسع في علوم الحاسبات وعلوم اخرى.

1.Propositions or Statements : العبارات

Definition: A Statements is a declarative sentence which is either (true: T) or (false: F) , but not both. We use the letters *p, q, r, s, …etc* to denote a Statements.

العبارة هي جملة خبرية والتي قد تكون صادقة أو كاذبة و من غير الممكن أن تكون العبارة صادقة وكاذبة بنفس الوقت .

Example: Which of the following sentences are called statements , and which ones are not statements.

1) p: $\sqrt{4} = 2$. Is true a statements.

2) q: $\sum_{x=1}^{3} (x + 2) = 13$. Is a false statements

Because $\sum_{x=1}^{3} (x + 2) = (1 + 2) + (2 + 2) + (3 + 2) = 3 + 4 + 5 = 12 \neq 13$.

3) r: Baghdad isn't in Iraq. Is a false statement

4) s: What time is it ? is not a statements.

5) $w: x + y = 0$. Is not a statements.

Example: (H. W) Which of the following sentences is called a statement, and which one is not a statement.

i) $p: x + 1 = 3$.

ii) $q: x + y = z$.

iii) r: 3/4 is an even number .

Definition: Negation of a proposition نفي العبارة

Let p be a proposition. The negation of p is called (not p) and is

denoted by $(\sim p)$.

Example: $\sim(3 < 5)$, $\sim(y > z)$, $\sim(y \geq 5)$, $\sim(2 = 10)$

Example: Find the truth value of each of the following statements. Find

$(\sim q)$ negations for the statements q and r.

1.p : Today is Saturday (F) , Today is not Saturday

2.q : 2+2=4 (T) $\sim q$:$2 + 2 \neq 4$

3. r: : The square has four sides (H. W)

Remark:

1. The truth table of the negation of a statements p

The truth table of the negation of a statements p	
p	$\sim p$
T F	F T

2. Double Negation Law: If p is a statements, then $\sim\sim p = p$.

2. Compound propositions العبارات المركبة

Statements are divided into two types:

1. Primitive Statements : A Statements is said to be Primitive Statements , if it cannot be divided into simpler Statements.
   تسمى العبارة بدائية او بسيطة إذا لم يمكن تحليلها إلى عبارات أبسط.
2. Composite Statements : A Statements is said to be Composite Statements , if it is compound of more than one primitive propositions using logical connective operators.
   تسمى العبارة مركبة إذا كانت تتكون من عبارتين بسيطتين أو أكثر تربطها أداة ربط واحدة او اكثر .

Basic Logical connective Operators **أدوات الربط المنطقية الاساسية** There are some basic logical operators that connect simple propositions to produce composite proposition. These operators are:

1. **Conjunction operator : (و) أداة الوصل** : English (and) , symbol (∧).

Let p and q are two primitive propositions. The conjunction of p and q is denoted by $(p \wedge q)$ and read as (p and q).
If both p and q are true, then $p \wedge q$ is true, otherwise $p \wedge q$ is false.

Below is the truth table for the conjunction of two propositions:

Conjunction		
p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Examples: Find the truth value of the following statements:

1. 2 + 2 = 4 and 2 + 3 = 5
   $T \quad \wedge \quad T \quad = T$

2. $x/x = 1$ such that $x \neq 0 \wedge$ Baghdad is not in Iraq.
   $T \quad \wedge \quad F \quad = F$
3. -5 is a prime number ∧ π is a rational number. (H. W)

....

Properties of the conjunction operators: (∧) خواص أداة الوصل

Let $p, q\ and\ r$ are three propositions. Using the truth table, show that:

1. $p \wedge q = q \wedge p$ (commutative) خاصية الإبدال

2. $(p \wedge q) \wedge r = p \wedge (q \wedge r)$ (associative) خاصية التجميع . (H. W)

3. $p \wedge p = p$ (Idempotent law) قانون تساوي القوى

4. $p \wedge T = p$ (Identity law) (H. W)

5. $p \wedge F = F$ (Domination Law)

6. $p \wedge \sim p = F$. (H. W)

Solution:

1.$p \wedge q = q \wedge p$

p	q	$p \wedge q$	$q \wedge p$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	F	F

3.$p \wedge p = p$

p	p	$p \wedge p$
T	T	T
F	F	F

5.$p \wedge F = F$

p	F	$p \wedge F$
T	F	F
F	F	F

2. **Disjunction operator (أداة الفصل (او):** English (and) , symbol ($\vee$).

Let p and q be two propositions. The disjunction of p and q is denoted by $(p \vee q)$ and read $(p\ or\ q)$.

We say that $(p \vee q)$ is true when p is true or q is true or both are true. If both p and q are false, then $p \vee q$ is false.

Below is the truth table for the disjunction of two propositions:

Disjunction		
p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Example: Let p, q and r are three propositions such that

p: dogs can fly

q: $x - x = 0, x \in R$

r: $-3 \in N$

Find the truth value of the following statements:

a) $(p \vee q) \vee r$ (H. W)

b) $\sim q \vee r$ (H. W)

c) $\sim(\sim p \vee q)$

d) $(p \wedge q) \vee (q \vee r)$ (H. W)

Solution of (c):

$\sim(\sim p \vee q) = \sim(T \vee T) = \sim T = F.$

Properties of the disjunction operator: ($\vee$) خواص أداة الفصل

Let p, q and r are three propositions. Using the truth table, show that:

1. $p \lor q = q \lor p$ (خاصية الابدال) (W .H)

2. $(p \lor q) \lor r = p \lor (q \lor r)$ (خاصية التجميع)

3. $p \lor p = p$ (قانون تساوي القوى) (W .H)

4. $p \lor T = T$ (Domination Law) (H. W)

5. $p \lor F = p$ (Identity Law) (H. W)

6. $p \lor \sim p = T$ (H. W)

<u>Solution2:</u> $(p \lor q) \lor r = p \lor (q \lor r)$

p	q	r	$p \lor q$	$q \lor r$	$(p \lor q) \lor r$	$p \lor (q \lor r)$
T	T	T	T	T	T	T
F	F	F	F	F	F	F
F	T	T	T	T	T	T
F	T	F	T	T	T	T
F	F	T	F	T	T	T
T	F	F	T	F	T	T
T	T	F	T	T	T	T
T	F	T	T	T	T	T

3. **Conditional operator** **أداة الشرط** : English word (if … then), Arabic word (اذا كان ... فأن...) , symbol (→)

Let p and q be two propositions. The conditional statement $p \longrightarrow q$ is the proposition $(if\ p\ then\ q)$. The conditional statement $p \longrightarrow q$ is false when p is true and q is false, otherwise $p \longrightarrow q$ is true.

The following is the truth table:

p	q	$p \longrightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

<u>Example</u>: Find the truth value of the following statement:

If fish fly, then 3 + 2 = 5

$F \quad \rightarrow \quad T \quad = T$

Example: Let p, q, and r are three propositions such that
p: 3 is an odd number
$q: x + y = y + x, x, y \in R$
r: Winter is hot
Find the truth value of the following statements:
1) $(p \to q) \lor (r \to q)$ (H. W)
2) $if\ (p \land q)\ then\ (q \lor \sim r)$
3) $(p \land r) \lor (q \to p)$ (H. W)
Solution2:
$if\ (p \land q)\ then\ (q \lor \sim r) = if\ (T \land T)\ then\ (T \lor T) = T \to T = T.$

Properties of the conditional operator: (→) خواص أداة الشرط

Let p, q and r are three propositions. Using the truth table show that: (H.W)
1. $(p \to q) \neq (q \to p)$
2. $(p \to q) \to r \neq p \to (q \to r)$
3. Find the truth value of: $p \to T$, $p \to F$, $p \to \sim p$, $p \to p$.

4. Bi-conditional operator أداة الشرط المزدوج :

English word (if and only if), Arabic word (.اذا وفقط اذا) , symbol (↔).
Let p and q be propositions. The bi-conditional statement $(p \leftrightarrow q)$ is the proposition ''p if and only if q''. The bi-conditional statement is true when p and q have the same true value, and is false otherwise.

The following is the truth table:

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Example: Find the truth value of $x > 0 \leftrightarrow 2x > 0$.
Solution: The statement is true because
$If\ x > 0\ then\ 2x > 0$ and $if\ 2x > 0\ then\ x > 0$
Example: Find the truth value of $x > 0 \leftrightarrow x^2 > 0$ (H. W.)

Properties of the biconditional operator :

Let p, q and r are three propositions. Using the truth table show that:

(H.W)

1. $p \leftrightarrow q = q \leftrightarrow p$
2. $(p \leftrightarrow q) \leftrightarrow r = p \leftrightarrow (q \leftrightarrow r)$
3. Find the truth value of: $p \leftrightarrow T, p \leftrightarrow F, p \leftrightarrow \sim p$, $p \leftrightarrow p$.

Definition: A compound proposition that is always true is called a tautology or lemma or theorem.

يقال للعبارة المركبة التي تكون صادقة دائما بأنها نظرية أو تحصيل حاصل.

A compound proposition that is always false is called a contradiction.

يقال للعبارة المركبة والتي تكون خاطئة دائما بأنها تناقض.

Example: Show that $(p \vee \sim p)$ is tautology and $(q \wedge \sim q)$ is contradiction.

Solution:

p	$\sim p$	$(p \vee \sim p)$ Tautology	$(q \wedge \sim q)$ contradiction
T F	F T	T T	F F

Example: (H. W) which of the following compound statements is tautology and which one is contradiction

$p \wedge F$, $p \vee T$, $p \leftrightarrow \sim p$, $[(p \rightarrow q) \wedge p] \wedge \sim p$.

Definition: Logical Equivalence التكافؤ المنطقي

Two statements (propositions) that have same truth values are called logically equivalent. The notation $p \equiv q$ or $p = q$ denotes that p and q are logically equivalent.

Example: show that $\sim(p \vee q) = \sim p \wedge \sim q$ (logically equivalent).

Solution: The truth table for $\sim(p \vee q)$ and $\sim p \wedge \sim q$ is

p	q	$p \vee q$	$\sim(p \vee q)$	$\sim p$	$\sim q$	$\sim p \wedge \sim q$
T T F F	T F T F	T T T F	F F F T	F F T T	F T F T	F F F T

Definition: Let p, q and r three propositions, then define the following logical equivalence:

1. $q \wedge r = \sim(\sim q \vee \sim r)$

2. $p \rightarrow q = \sim p \vee q$

3. $p \leftrightarrow q = (p \rightarrow q) \wedge (q \rightarrow p)$

De Morgan's Theorem: Let p and q are two propositions. Then

1. $\sim(p \wedge r) = \sim p \vee \sim q$ (H. W)

2. $\sim(p \vee r) = \sim p \wedge \sim q$.

Proof (2): Take the right hand side (R. H. S)

$\sim p \wedge \sim q = \sim(\sim\sim p \vee \sim\sim q)$ [definition of $\wedge$]

$= \sim(p \vee q)$ [double negation law: $\sim\sim p = p$]

= Left hand side (L. H. S).

Exercise: Simplify the following statements:

1. $\sim(p \vee \sim q)$

2. $\sim(\sim p \rightarrow q)$

3. $\sim(\sim p \leftrightarrow q)$ (H.W)

Solution(1): $\sim(p \vee \sim q) = \sim p \wedge \sim\sim q$ [De Morgan's law]

$= \sim p \wedge q$ [$\sim\sim q = q$]

Solution(2): $\sim(\sim p \rightarrow q) = \sim(\sim\sim p \vee q)$

$= \sim(p \vee q)$ [$\sim\sim p = p$]

$= \sim p \wedge \sim q$ [De Morgan's law]

Laws of Logical Equivalence قوانين التطابق المنطقي

Let p, q and r are propositions. The following are some of the common logical equivalence rules:

1. Distributive Law (from left) قانون التوزيع من اليسار :

- $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$
- $p \wedge (q \wedge r) = (p \wedge q) \wedge (p \wedge r)$
- $p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$
- $p \vee (q \vee r) = (p \vee q) \vee (p \vee r)$
- $p \vee (q \rightarrow r) = (p \vee q) \rightarrow (p \vee r)$
- $p \vee (q \leftrightarrow r) = (p \vee q) \leftrightarrow (p \vee r)$

2. Distributive Law (from right) :قانون التوزيع من اليمني

- $(q \vee r) \wedge p = (q \wedge p) \vee (r \wedge p)$
- $(q \wedge r) \wedge p = (q \wedge p) \wedge (r \wedge p)$
- $(q \wedge r) \vee p = (q \vee p) \wedge (r \vee p)$
- $(q \vee r) \vee p = (q \vee p) \vee (r \vee p)$
- $(q \rightarrow r) \vee p = (q \vee p) \rightarrow (r \vee p)$
- $(q \leftrightarrow r) \vee p = (q \vee p) \leftrightarrow (r \vee p)$

Exercise: Simplify the following statements using laws of logical equivalence: (H.W)

1. $(p \vee q) \wedge \sim p$.

2. $(p \vee q) \vee (\sim p \wedge q)$.

Exercise: Prove that (without using the truth table)

$\sim(p \vee (\sim p \wedge q)) = \sim p \wedge \sim q$.

Solution: Take the L. H. S

$\sim(p \vee (\sim p \wedge q)) = \sim p \wedge \sim(\sim p \wedge q)$ [De Morgan’s law]

$= \sim p \wedge (\sim\sim p \vee \sim q)$ [De Morgan’s law]

$= \sim p \wedge (p \vee \sim q)$ [by double negation law]

$= (\sim p \wedge p) \vee (\sim p \wedge \sim q)$ [by distributive law]

$= F \vee (\sim p \wedge \sim q)$ [$p \wedge \sim p = F$]

$= (\sim p \wedge \sim q) \vee F$ [by commutative law]

$= \sim p \wedge \sim q$ R. H. S

Theorem: (Properties of →)

Let p, q and r are three propositions. Prove the following properties without using truth tables:

1. $p \to p = T$
2. $\sim p \to p = p$
3. $p \to T = T$
4. $T \to p = p$
5. $p \to F = \sim p$
6. $F \to p = T$
7. $p \to q = \sim q \to \sim p$
8. $p \to q = (p \wedge \sim q) \to \sim p$
9. $p \to q = (p \wedge \sim q) \to (r \wedge \sim r)$
10. $\sim(p \to q) = p \wedge \sim q$

Proof 1: To prove $p \to p = T$

$p \to p = \sim p \vee p$ [def. of →]

$= T$

Proof 4: To prove $T \to p = p$

$T \rightarrow p = \sim T \vee p$ [def. of →]

$= F \vee p = p$

Proof 7: To prove $p \rightarrow q = \sim q \rightarrow \sim p$

$p \rightarrow q = \sim p \vee q$ [def. of →]

$= q \vee \sim p$ [∨ is commutative]

$= \sim\sim q \vee \sim p = \sim q \rightarrow \sim p$

Proof 8: To prove $p \rightarrow q = (p \wedge \sim q) \rightarrow \sim p$

Take the R. H. S: $(p \wedge \sim q) \rightarrow \sim p$

$= \sim(p \wedge \sim q) \vee \sim p$ [def. of →]

$= (\sim p \wedge \sim\sim q) \vee \sim p$ [De Morgan]

$= (\sim p \vee q) \vee \sim p$ [$\sim\sim q = q$]

$= \sim p \vee (\sim p \vee q)$ [∨ is comm.]

$= (\sim p \vee \sim p) \vee q$ [∨ is associative]

$= \sim p \vee q$ [$p \vee p = p$]

$= p \rightarrow q$ [def. of →]

= L. H. S

Theorem: (Properties of ↔)

Let p and q are two propositions. Prove the following properties without using truth tables:

1. $p \leftrightarrow p = T, \quad p \leftrightarrow T = p, \quad p \leftrightarrow F = \sim p$

2. $p \leftrightarrow \sim p = F$

3. $p \leftrightarrow q = q \leftrightarrow p$

4. $p \leftrightarrow q = \sim p \leftrightarrow \sim q$

5. $\sim p \leftrightarrow q = p \leftrightarrow \sim q$

6. $\sim (p \leftrightarrow q) = \sim p \leftrightarrow q$

7. $\sim (p \leftrightarrow q) = p \leftrightarrow \sim q$

Proof 1: To prove $p \leftrightarrow T = p$

$p \leftrightarrow T = (p \rightarrow T) \wedge (T \rightarrow p) =$ [def. of $\leftrightarrow$]

$= (\sim p \vee T) \wedge (\sim T \vee p)$ [def. of]

$= (\sim p \vee T) \wedge (F \vee p)$ [$\sim T = F$]

$= T \wedge p$ [$\sim p \vee T = T$]

$= p$

Proof 6: $\sim (p \leftrightarrow q) = \sim p \leftrightarrow q$

Take L. H. S : $\sim (p \leftrightarrow q) = \sim [(p \rightarrow q) \wedge (q \rightarrow p)]$ [def. of $\leftrightarrow$]

$= \sim (p \rightarrow q) \vee \sim (q \rightarrow p)$ [De Morgan]

$= \sim (\sim p \vee q) \vee \sim (\sim q \vee p)$ [def. of $\rightarrow$]

$= (p \wedge \sim q) \vee (q \wedge \sim p)$ [De Morgan]

$= [(p \wedge \sim q) \vee q] \wedge [(p \wedge \sim q) \vee \sim p)]$ [distributive ($\vee$ on $\wedge$)]

$=[(p \vee q) \wedge (\sim q \vee q)] \wedge [(p \vee \sim p) \wedge (\sim q \vee \sim p)]$ [dist. ($\vee$ on $\wedge$)]

$= [(p \vee q) \wedge T] \wedge [T \wedge (\sim q \vee \sim p)]$

$= (p \vee q) \wedge (\sim q \vee \sim p)$

$= (\sim p \rightarrow q) \wedge (q \rightarrow \sim p)$ [def. of $\rightarrow$]

$= \sim p \leftrightarrow q$ [def. of $\leftrightarrow$]

Mathematical Proof البرهان الرياضي

A mathematical proof is a valid argument that establish the truth of a mathematical statement.

البرهان الرياضي هو إثبات صحة عبارة رياضية من خلال حجة أو تعليل منطقي.

Methods of Proving Mathematical Statements (or Theorems):

1. Direct Proof of a conditional statement $q \rightarrow r$:

Direct proofs lead from the hypothesis of a theorem to the conclusion.

Definition: The integer number x is called even if there exist $k \in Z$ such that $x = 2k$.

Definition: The integer number x is called odd if there exist $k \in Z$ suchthat $x = 2k + 1$.

Theorem: If x is an odd natural number ($x \in O$) then x^2 is odd.

Proof: Assume that x is an odd natural number.

We must prove x^2 is odd,

Since x is odd, then $x = 2k + 1$ for some $k \in N$.

$x^2 = x.x = (2k + 1)(2k + 1) = 4k^2 + 4k + 1$

$= 2(2k^2 + 2k) + 1$

Let $s = 2k^2 + 2k \in O$, then $x^2 = 2s + 1$

Hence, x^2 is an odd number.

Theorem: (H. W.) If x is an even natural number ($x \in E$) then x^2 is even.

Theorem: The sum of two even natural numbers is even.

The theorem can be written as follows: $If\ x, y \in E^{+}\ then\ x + y \in E^{+}$

where $E^{+} =$ set of positive even numbers.

Proof: Let p: x and y are even positive numbers,

q: $x + y$ is an even positive number

Let $x = 2r$ and $y = 2s$ ($s, t \in N$).

Then $x + y = 2r + 2s = 2(r + s)$ such that $s + t \in N$

$x + y = 2k\ where\ k = s + t.$

Therefore $x + y$ is a positive even number.

Theorem: (H. W.)

i) If $x \in E$ and $y \in O$ then $x + y \in O$

ii) If $x \in E$ and $y \in O$ then $x.y \in E$

iii) If $x, y \in E$ then $x + y \in E$

2. Direct Proof of a conditional statement p ↔ q

To prove a proposition in the form p ↔ q, we prove its equivalence. i.e.,

$p \leftrightarrow q = (p \rightarrow q) \wedge (q \rightarrow p).$

Theorem: x is odd number ↔ $x + 1$ is an even number

Proof: Let p: x is odd number

q: $x + 1$ is an even number

1. Prove p →q: Let $x \in O, x = 2k + 1 ; k \in Z$

$x + 1 = 2k + 2 = 2(k + 1); (k + 1) \in Z$

$x + 1 = 2r ; r = k + 1 \in Z$

$x + 1 \in E$

2. Prove q → p: Let $x + 1 \in E$ To prove $x \in O$

$x + 1 = 2k$; $k \in Z$

$x = 2k - 1$; $k \in Z$

$x = 2k - (2-1) = 2k - 2 + 1 = 2(k-1) + 1$

Since $(k-1) \in Z$, then $x = 2r + 1, r = k - 1 \in Z$

$x = 2r + 1 \in O$

Theorem: (H.W) x is even $\Leftrightarrow x^2$ is even .

Theorem: (H. W.) x is odd number **if and only if** x^2 is odd number.

3. Proof by Contradiction:

Theorem: Prove that: If $x^2 \in O$ then $x \in O$

Proof: Assume that $x^2 \in O$. To prove $x \in O$

By contradiction, assume that $x \in E$

$x = 2k$; $k \in Z$

$x^2 = 4k^2 \in E$ تناقض مع الفرض

$\therefore x \in O$.

Theorem: If x^2 is even then x is even.

Proof: Assume that $x^2 \in E$. To prove $x \in E$

By contradiction, assume that $x \in O$

$x = 2k + 1$; $k \in Z$

$x^2 = 4k^2 + 4k + 1 \in O$ تناقض مع الفرض

$x \in E$.

<u>Definition:</u> Variable المتغير

An alphabetic letter $x, y, z, \ldots$ which represents a number that is either arbitrary or unknown.

<u>Example</u>: $4x - 7 = 5$: x is a variable

$\sqrt[3]{z} = 3$: z is a variable

<u>Definition</u>: Open Sentence الجملة المفتوحة

A sentence is called open sentence (or propositional function), if it contains one or more variables. Open sentence is denoted by $p(x), q(x), g(x) \ldots etc.$

<u>Example:</u> The following are open sentences:

$p(x)$: x is an odd number

$q(x,y)$: $x + y = 5$ such that $x, y \in N$

$r(z)$: $\sqrt[3]{z} = 3$ such that $z \in R$

<u>Definition</u>: Solution Set (Truth set)

Let $p(x)$ be an open sentence and let A be a set. The solution set denoted by T_p is the set of all elements x of A for which $p(x)$ is true. In other words $T_p = \{x \in A : p(x) \text{ is true}\}$

مجموعة الحل أو مجموعة الصدق : هي مجموعة العناصر التي تجعل التعبير المفتوح $p(x)$ عبارة صادقة .

<u>Example:</u> Find the solution set for of the following open sentence:

Let $p(x)$ be $(x + 2 > 7)$ and $A = N$. Then

$T_p = \{x \in N: x + 2 > 7\} = \{x \in N : x > 5\} = \{6, 7, \ldots\}$

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Quantifiers : المسورات

Quantifiers are open sentences written in a special way.

المسورات هي جمل مفتوحة مكتوبة بطريقة معينة.

There are two types of quantifiers:

1. Universal quantifiers العبارة المسورة كليا العبارة

2. Existential quantifiers المسورة جزئيا

Universal quantifiers:

Let $p(x)$ be an open sentence on a set A. The notation ($\forall\, x\ \in\ A, p(x)$)

Denote the universal quantification تسوير كلي of $p(x)$ and it reads as:

For all $x, p(x)$'' or ''for every $x, p(x)$'' or ''for each $x, p(x)$''.

The symbol ∀ is called universal quantifiers.

The set A is called domain . المجال

Example: $\forall\, x\ \in\ N\ ,\ x\ >\ 0$

All seasons in Iraq have rain.

Remark : 1. The universal quantifier $p(x)$ on a domain A is true if and only if $T_p = A$.

2. universal quantifier $p(x)$ on a domain A is false if and only if there exist $x \in\ A$ such that $p(x)$ is false.

Example: Find the truth value of the following open sentences:

1. $\forall\, x\ \in R\ , x\ +\ 1\ >\ x$

Let $A\ =\ R\ and\ p(x): x\ +\ 1\ >\ x$

Because $p(x)$ is true for all $x\ \in\ R$, the solution set $T_p = R$.

$\Longrightarrow$ the quantification $\forall\, x \in R,\ x + 1 > x$ is true.

2. $\forall\, x \in R, x < 2$

Let $A = N$ and $p(x): x < 2$

$p(x)$ is not true for all $x \in N$. Take $x = 3, p(3)$ is false.

$\Longrightarrow T_p \neq N$

3. $\forall\, x \in N, (x > 0\ and\ x = 0)$

The statement is false, there exists $x = 4 \in N$ such that $4 > 0$ and $4 \neq 0$.

4. $\forall\, x \in Z,\ |x| > 0$ (H. W.)

5. For all $x \in \{1, -1\}, x^2 - 1 = 0$ (H. W.)

Existential quantifiers:

Let $p(x)$ be an open sentence on a set A. The notation

$$\exists\, x \in A, P(x)$$

Denote the existential quantification تسوير جزئي of $p(x)$ and it reads as:

"there exists x, $p(x)$" or "there is x, $p(x)$" or "some x, $p(x)$".

The symbol ∃ is called existential quantifier مسور جزئي

The set A is called domain . المجال

Example: $\exists\, x \in N, x < 0$

There exists seasons in Iraq do not have rain

Remark:

1.The existential quantifier $p(x)$ on a domain A is true if and only if $T_p \neq \emptyset$.

العبارة المسورة جزئيا تكون صادقة إذا وجد على الاقل عنصر واحد يحقق العبارة$p(x)$.

The existential quantifier $p(x)$ on a domain A is false if and only if $T_p = \emptyset$.

Example: Find the truth value of the following open sentences:

1. $\exists x \in R,\ x^2 = x$

$A = R \text{ and } p(x):\ x^2 = x$

$T_p = \{0,1\}$

$\Longrightarrow$ the existential quantifier $\exists x \in R,\ x^2 = x$ is true.

2. $\exists x \in N, 3x + 5 = 1$

$x = -4/3 \notin N \Longrightarrow T_p = \emptyset$

$\Longrightarrow x \in N, 3x + 5 = 1$ is false

3. $\exists x \in Z, [(x + 1)^2 = 0 \text{ and } x^2 - 1 = 0]$

$(x + 1)^2 = 0 \Longrightarrow x = -1$

And $x^2 - 1 = 0 \Longrightarrow x = -1, 1$

$T_p = \{-1\} \subset Z$

$\exists x \in Z, [(x + 1)^2 = 0 \text{ and } x^2 - 1 = 0]$ is true

De Morgan's law for the existential quantifier

$\sim [\exists\, x \in A, \sim p(x)] = \forall x \in A, p(x)$

قانون دي موركان للعلاقة بين التسوير الكلي والجزئي

Example:

$\sim [\exists\, x \in E, x + 2 \notin F] = \forall x \in E, x + 2 \in E$

Theorem: Let $p(x)$ be an open sentence and A is the domain. Then

1. $\sim[\forall x \in A, p(x)] = \exists\, x \in A, \sim p(x)$

2. $\sim[\forall x \in A, \sim p(x)] = \exists x \in A, p(x)$ (H. W.)

3. $\sim[\exists x \in A, p(x)] = \forall x \in A, \sim p(x)$ (H. W.)

Proof1: $\sim[\forall x \in A, p(x)] = \sim[\sim[\exists x \in A, \sim p(x)]]$ {from De Morgan}

$= \sim\sim[\exists x \in A, \sim p(x)]$

$= \exists x \in A, \sim p(x)$ $[\sim\sim p = p]$

End