Chapter Three: Relations العلاقات

Chapter Three Contents:

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<u>Definition:</u> Ordered Pair الزوج المرتب

An **ordered pair** of elements a and b *is* denoted by (a, b) where a is called the first element and b is the second element.

ليكن a و b عنصراً . الزوج المرتب المتكون من a و b يرمز له بالرمز (a, b) حيث أن a يسمى مسقط أول للزوج المرتب والعنصر b يسمى المسقط الثاني.

<u>Remark:</u> Let a, b, c and d be four elements. Then:

1. $(a,b) \neq (c,d)$ In general
2. $(a,b) = (c,d) \Leftrightarrow a = c \wedge b = d$
3. $(a,b) = (b,a) \Leftrightarrow a = b$

<u>Cartesian Product</u> الضرب الديكارتي

Consider two arbitrary sets A and B. The set of all ordered pairs (a, b) where $a \in A$ and $b \in B$ is called the **product**, or **Cartesian product**, of A and B and denoted by $(A \times B)$.

$$A \times B = \{(a,b): a \in A \wedge b \in B\}$$
$$(a,b) \in A \times B \Leftrightarrow a \in A \wedge b \in B$$
$$(a,b) \notin A \times B \Leftrightarrow a \notin A \vee b \notin B$$

Example3.3: Let $A = \{a,b,c\}, B = \{5,4\}$. Find

$A \times B = \{(a,5),(a,4),(b,5),(b,4),(c,5),(c,4)\}$

$B \times A =$

$A \times A =$

$B \times B =$

<u>Remark:</u> If the number of a set A equals n and the number of a set B equals m. Then the number of the elements of $A \times B$ is nm.

Example3.5: Let $A = \{x \in N: x \leq 3\} = \{1,2,3\}$, $B = \{0,3\}$ and $C = \{1\}$ find :

$A \times A =$
$B \times B =$
$C \times C =$
$B \times C =$
$(B \cap C) \times A =$
$(B \cup C) \times B =$
$Is, A \times B = B \times A?$

<u>Theorem:</u> Let A, B, C and D be nonempty sets. Then:

1. $A \times \emptyset = \emptyset$ and $\emptyset \times A = \emptyset$
2. $A \times B = B \times A \iff A = B$
3. $A \times (B \cap C) = (A \times B) \cap (A \times C)$. **(H. W.)**
4. $A \times (B \cup C) = (A \times B) \cup (A \times C)$. **(H. W.)**
5. $A \times (B \backslash C) = (A \times B) \backslash (A \times C)$
6. $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$ **(H. W.)**

Proof 1: T. P. $\emptyset \times A = \emptyset$
Suppose that $\emptyset \times A \neq \emptyset$ نفرض العكس
$\exists\ (x, y) \in \emptyset \times A \implies x \in \emptyset\ \wedge\ y \in A$ (def. of $A \times B$)
$\implies F\ \wedge\ y \in A$
$\implies\ F\ (F\ \wedge\ p = F)$ تناقض
$\therefore \emptyset \times A = \emptyset$
In the same way, prove that $A \times \emptyset = \emptyset$ (**H. W**)

Proof 2:

($\implies$) Suppose $A \times B = B \times A$ T. P. $A = B$

Let $x \in A \wedge y \in B$

$\implies (x, y) \in A \times B$ (def. of $A \times B$)

$\implies (x, y) \in B \times A$, $(A \times B = B \times A)$

$\implies x \in B \wedge y \in A$

$\implies A \subseteq B \wedge B \subseteq A$

$\implies A = B$ (def. of equal sets)

($\Leftarrow$) Suppose $A = B$ T. P. $A \times B = B \times A$

Let $(x, y) \in A \times B$

$\Leftrightarrow x \in A \wedge y \in B$ $(def. of\ A \times B)$

$\Leftrightarrow x \in B \wedge y \in A$ $(A = B)$

$\Leftrightarrow (x, y) \in B \times A$

$\therefore A \times B = B \times A$

Proof 5:

T. P. $A \times (B \backslash C) = (A \times B) \backslash (A \times C)$
Let $(x, y) \in A \times (B \backslash C)$
$\Leftrightarrow x \in A \wedge y \in (B \backslash C)$ (def. of $A \times B$)
$\Leftrightarrow x \in A \wedge (y \in B \wedge y \notin C)$ (def. of \)
$\Leftrightarrow (x \in A \wedge y \in B) \wedge (x \in A \wedge y \notin C)$ (dist. $\wedge$ on $\wedge$)
$\Leftrightarrow (x, y) \in A \times B \wedge (x, y) \notin A \times C$
$\Leftrightarrow (x, y) \in (A \times B) \backslash (A \times C)$
$\therefore A \times (B \backslash C) = (A \times B) \backslash (A \times C)$

Definition: Generalization of the Cartesian product تعميم الضرب الديكارتي
Let $A_1, A_2, \ldots, A_n$ be any sets. Then

$$\prod_{i=1}^{n} A_i = A_1 \times A_2 \times \ldots \times A_n = \{(x_1, \ldots, x_n) : x_i \in A_i, i = 1, \ldots, n\}$$

Example: What is the Cartesian product $A \times B \times C$, where
$A = \{0,1\}, B = \{1,2\}, C = \{2\}$?

Solution: $A \times B \times C = \{(0,1,2), (0,2,2), (1,1,2), (1,2,2)\}$

Remark: Let A be a set, then $A \times A = A^2$ and $A \times A \times A = A^3$
In general, $A \times \ldots \times A = A^n$

Example: $R \times R = R^2$ and $R \times R \times R = R^3$

Relation: العلاقة

Let A and B are two sets. Any subset R of A $\times$ B is called a relation from A to B. In other words,
R is a relation from A to B $\Leftrightarrow$ R $\subseteq$ A $\times$ B.

$(\mathbf{x},\mathbf{y}) \in$ **R can be written as xRy or x~y**
$(\mathbf{x},\mathbf{y}) \notin$ **R can be written as xRy or** $\boldsymbol{x} \nsim \boldsymbol{y}$

Remark: The relations are denoted by $R, S, T, W, \ldots$

Definition: If R is a relation from $A\ to\ A$ ($R \subseteq A \times A$) then R is called a relation on A.

Example: Let $A = \{1, 4, 5\}$ and $B = \{1, a\}$
$A \times B = \{(1,1), (1, a), (4,1), (4, a), (5,1), (5, a)\}$. Write three relations from $A\ to\ B$.

Solution:

$R =$

$S =$

$T =$

Remark: The empty set $\emptyset$ is the smallest relation from A to B ($\emptyset \subseteq A \times B$). And, $A \times B$ is the largest relation from A to B ($A \times B \subseteq A \times B$)

Example: (H. W.) Let $A = \{1, 4, 5\}$, $B = \{1, a\}$. Find $B \times A$ and write three relations from $B\ to\ A$.

Example: Let $A = \{x, y, -1\}$. Find a relation from A to A.

Solution: $A \times A = \{(a, b): a, b \in A\}$
$A \times A = \{(x, x), (x, y), (x, -1), (y, x), (y, y), (y, -1), (-1, x), (-1, y), (-1, -1)\}$
then $R = \{(x, x), (x, y), (x, -1), \}$ be a relation from A to A.

<u>Specifying a relation:</u> طرق التعبير عن العلاقة

<u>1 .Listing members of a relation</u>: الطريقة الجدولية

List the members of the relation separated by commas and contained in braces { }.

Example: Let A $= \{1,2\}$ and B $= \{x, y\}$ are two sets and R is a relation from A B. R $= \{(1, x), (1, y), (2, x), (2, y)\}$

2. Listing a relation property: الصفة المميزة للعلاقة

State the property that characterizes the elements in a relation
Example: $A = \{-1,5,0\}$ and $= \{-3,0,-1\}$.
Let $R_1 = \{(a, b) \in A \times B: a \geq b\}$
$= \{(-1,-3), (-1,-1), (5,-3), (5,0), (5,-1), (0,-3), (0,0), (0,-1)\}$
$R_2 = \{(x, y) \in A \times B: x = y\} = \{(-1,-1), (0,0)\}$

Definition: Let x *and* y are integers with $x \neq 0$ $(x, y \in Z)$. Then "x divides y" يقسم is denoted by $x|y$ and is defined as:

$$\boldsymbol{x|y \Leftrightarrow \exists k \in Z \text{ such that } y = kx}$$
$$\boldsymbol{x \nmid y \Leftrightarrow \forall k \in Z \text{ such that } y \neq kx}$$

When "x divides y" يقسم we say that "x is a factor of y" or "y is a multiple of x"

Example: Are $3|6, -4|8, -5|1, 9|30$?

Solution:

$\exists k = 2$ *such that* $6 = 3k = 3(2) \Longrightarrow 3|6$
$\exists k = -2$ *such that* $8 = -4k = -4(-2) \Longrightarrow -4|8$
$\forall k \in Z, 30 \neq 9k \Longrightarrow 9 \nmid 30$
$-5|1$ (H. W.)

Example: $A = \{x \in Z: 0 \leq x \leq 4\}$. Write a relation R on A such that $R = \{(a, b) \in A \times A: a|b\}$

Solution: $A = \{0,1,2,3,4\}$ and
$R =$
$\{(1,0), (1,1), (1,2), (1,3), (1,4), (2,0), (2,2), (2,4), (3,0), (3,3), (4,0), (4,4)\}$

Definition: Let R_1 and R_2 are two relations from A to B, then $R_1 \cap R_2$, $R_1 \cup R_2$, *and* $R_1 \backslash R_2$ are also relations from A *to* B.

Example: $A = \{x, y, z\}$ and $B = \{x \in Z: -2 \leq x \leq 3\}$ Write two relations R_1 and R_2 from A *to* B.

R_1=

R_2=

Then find

$R_1 \cap R_2 =$

$R_1 \cup R_2 =$

$R_1 \backslash R_2 =$

$R_2 \backslash R_2 =$

Definition: Let R a relation from A *to* B then R^{-1} is a relation from B *to* A. R^{-1} is called the inverse of R.

$$R^{-1} = \{(b, a): (a, b) \in R\}$$

Example: Let $= \{0,3,8,-10\}$ $B = \{0,1,2\}$. A relation R from A *to* B is defined as: $R = \{(a, b) \in A \times B: a + b = 2k, k \in Z\}$. Write the elements of the relation R and find R^{-1} ?

Solution:

$R = \{ (0,0), (0,2), (3,1), (8,0), (8,2), (-10,0), (-10, 2)\}$

$R^{-1} = \{ (0,0), (2,0), (1,3), (0,8), (2,8), (0, -10), (2, -10)\}$

Definition: Domain of a relation مجال العلاقة

The domain of a relation $R \subseteq A \times B$ is the set of the first coordinates of each pair. In other words:

$dom\ R = \{x \in A; \exists y \in B: (x, y) \in R\}$

It is clear that $dom\ R \subseteq A$ منطلق العلاقة هو مجموعة المساقط الاولى للعلاقة

Definition: Range of a relation مدى العلاقة

The range of a relation $R \subseteq A \times B$ is the set of the second coordinates of each pair. In other words:

$$\text{range } R = \{y \in B;\ \exists x \in A: (x, y) \in R\}$$

It is clear that range $R \subseteq B$ مدى العلاقة هو مجموعة المساقط الثانية للعلاقة

Example: Let $A = N$ $R = \{(a, b) \in N \times N: b = 2a\}$. Find $dom\, R$ $range\, R$.

Solution: The relation R can be written as follows
$R = \{(1,2), (2,4), (3,6), (4,8), \ldots .\}$
$DomR = \{1,2,3,4,5, \ldots \ldots\} = N$
$Range\ R = \{2,4,6,8, \ldots .\} = E^{+}$ even positive numbers

Example: (H. W.) Let $A = Z\ and\ R = \{(a, b) \in Z \times Z: b = 1 - a\}$
Find $dom\ R$ $range\ R$.

Lemma: Let R be a relation on $A \times B$ then:

1. $dom\ R = range\ R^{-1}$ (H. W.)
2. $range\ R = dom\ R^{-1}$

Proof 2:

$A \times B = \{(a, b): a \in A, b \in B\}$
$B \times A = \{(b, a): (a, b) \in A \times B\}$
$rangeR = \{b \in B;\ \exists a \in A\ such\ that\ (a, b) \in R\}$
$domR^{-1} = \{b \in B;\ \exists a \in A\ such\ that\ (b, a) \in R^{-1}\}$
T. P $range\ R = dom\ R^{-1}$
Let $b \in rangeR$
$\Leftrightarrow \exists a \in A\ such\ that\ (a, b) \in R$
$\Leftrightarrow \exists a \in A\ such\ that\ (b, a) \in R^{-1}$
$\Leftrightarrow b \in dom\ R^{-1}$
$\therefore range\ R = dom\ R^{-1}$

Properties of Relations: خواص العلاقات

1. Reflexive relation: العلاقة الانعكاسية

A relation R on a set A is called **reflexive** if $(a, a) \in R$ for every $a \in A$.

R reflexive on $A \Leftrightarrow a\ R\ a\ , \forall a \in A \Leftrightarrow (a, a) \in R,\ \forall a \in A$

R not reflexive on $A \Leftrightarrow \exists a \in A, a \nsim a \Leftrightarrow \exists a \in A,\ (a, a) \notin A$

Example: Let $A = \{1,2,3,4\}$. Which of these relations are reflexive?
Solution:

$R_1 = \{(1,1), (1,2), (2,1), (2,2), (3,3), (3,4), (4,1), (4,4)\}$
R_1 is reflexive on A, because
$(1,1) \in R_1, (2,2) \in R_1, (3,3) \in R_1, (4,4) \in R_1$
$\Longrightarrow (a, a) \in R_1\ \forall a \in A$

$R_2 = \{(1,1), (1,2), (2,1)\}$ is not reflexive because,
$\exists 2 \in A\ such\ that\ (2,2) \notin R_2$
$\exists 3 \in A\ s\quad ch\ that\ (3,3) \notin R_2$
$\exists 4 \in A\ such\ that\ (4,4) \notin R_2$

$R_3 = \{(1,1), (2,1), (3,4), (2,2), (2,3), (3,3), (3,1), (3,1), (4,3)\}$ is not reflexive because, $(4,4) \notin R_3$

$R_4 = \{(3,4)\}$ is not reflexive.

Example: Let $A = \{-2, 1/\ 2\ , 0, 3\}$. Let R_1 and R_2 be two relations on A such that:
$R_1 = \{(a, b): a \leq b\}$ and $R_2 = \{(a, b): a = b\}$.
Are $R_1\ and\ R_2$ reflexive? Is $A \times A$ reflexive on A?

Solution:

$R_1 = \{(a, b): a \leq b\} = \{(-2, -2), (-2, 1/2\ ), (-2,0), (-2,3), (0, 1/\ \underline{2}\ ), (0,3), (\ 1/2\ , 3)\ , (\ 1/2\ , 1/2\ )\ , (0,0), (3,3)\}$
$(a, a) \in R_1, \forall a \in A$
$\Longrightarrow R_1$ is reflexive

Similarly, $(a, a) \in R_2\ \forall a \in A \Longrightarrow R_2$ is reflexive relation

And $(a, a) \in A \times A\ \forall a \in A \Longrightarrow A \times A$ is reflexive relation

Example:
Let R be relations on Z such that $R = \{(a, b): a = b \text{ or } a = -b\}$. Is R reflexive? Is $Z \times Z$ reflexive on Z?

Solution:
$R = \{(-1, -1), (-1,1), (1, -1), (0,0), (2, -2), \ldots .\}$
Since $(a, a) \in R\ \forall a \in Z$
$\Longrightarrow R$ is reflexive

Similarly, $(a, a) \in Z \times Z\ \forall a \in Z \Longrightarrow Z \times Z$ is reflexive

Remarks:

1. $A \times A$ is reflexive on A
2. $\emptyset$ is not reflexive on A

Example: Let $A = N$. Let R be relations on A such that
$R = \{(a, b) \in N \times N: a|b\}$. Is R reflexive?

Solution:

Since $\exists k = 1$ s.t. $a = 1(a) \Longrightarrow a|a \Longrightarrow a\, R\, a\ \forall a \in N$
$\Longrightarrow R$ is reflexive

Example: Let $A = \{-2, -3, 2, 4\}$. Let R be relations on A such that
$R = \{(a, b): a + b \leq 3\}$. Is R reflexive?

Solution:
Let $a = 2$, $b = 2$ then $a + b = 4 > 3$
$\Longrightarrow R$ is not reflexive

Identity Relation : العلاقة الذاتية او علاقة الوحدة

Let A be a set.The identity relation on A is denoted by I_A and is defined as:

$$I_A = \{(a, b) \in A \times A: a = b\}$$

Remark: Let A be a set. The identity relation I_A is a subset of the reflexive relation on A.

Example: Let $A = \{-2, -3, 2, 4\}$. The following relation is Identity relation $R = \{(-2,-2), (-3,-3), (2,2), (4,4)\} = I_A$

Symmetric Relation العلاقة التناظرية

A relation R on a set A is called symmetric if the following condition satisfied:

$$\text{If } (a,b) \in R \text{ then } (b,a) \in R\ \forall a,b \in A$$

The relation R is not symmetric غير تناظرية if :

$$\exists (a,b) \in A \times A \text{ such that } (a,b) \in R \text{ but } (b,a) \notin R$$

Example: Let $A = \{1,2,3,4\}$. Which of these relations are symmetric?
$R_1 = \{(1,2), (2,1)\}$ is symmetric because $(1,2) \in R_1 \wedge (2,1) \in R_1$

$R_2 = \{(1,1), (1,2), (2,1)\}$ is symmetric

$R_3 = \{(1,1), (3,4), (2,2), (2,3), (3,3), (3,1), (1,3), (4,3)\}$ is not symmetric because $(2,3) \in R_3$ but $(3,2) \notin R_3$

$R_4 = \{(3,4)\}$ is not symmetric

Example: Let $A = \{-2, -3, 2, 4\}$. Let R be relations on A such that $R = \{(a,b): a + b \leq 3\}$. Is R symmetric?

Solution:
$R = \{(-2,-2), (-2,-3), (-2,2), (-2,4), (-3,-2), (-3,2), (-3,4),$

$(2,-2), (2,-3), (4,-2), (4,-3)\}$ R is symmetric,
$(a,b) \in R \Leftrightarrow (b,a) \in R$

Example: Let $A = Z$. Let R be relations on A such that $R = \{(a,b) \in Z \times Z: a + b \leq 3\}$. Is R symmetric?

Solution: $R = \{(-1,-1), (-2,-2), (2,1), (1,2), (1,1), \dots .\}$

نلاحظ في هذا المثال ان العلاقة R معرفة على مجموعة الاعداد الصحيحة الغير منتهية وانه من غير الممكن كتابة جميع عناصر العلاقة R لذلك فانه من الصعب اختبار خاصية التناظر من خلال كتابة كل عناصر العلاقة. لذلك سنلجأ الى طريقة الاختبار عن طريق البرهان العام كالتالي:

Let $(a, b) \in R \Longrightarrow (b, a) \in R$?
$(a, b) \in R \Longrightarrow a + b \leq 3 \Longrightarrow b + a \leq 3 \Longrightarrow (b, a) \in R$
$\therefore$ R is symmetric relation

Example: Let $A = Z$ and define $a R b \Leftrightarrow ab \geq 0 \ \forall a, b \in Z$.
Is R reflexive? Symmetric?

Solution: reflexive: Is $aRa \ \forall a \in Z$? Let $a \in Z \Longrightarrow a.a = a^2 \geq 0$
$\therefore R$ is reflexive

symmetric? Let $(a, b) \in R$, $(b, a) \in R$?
$(a, b) \in R \Longrightarrow ab \geq 0 \Longrightarrow ba \geq 0 \Longrightarrow (b, a) \in R$
$\therefore R$ is symmetric

Example: Let $A = R$ and define $S\, b \Leftrightarrow a - b > 0 \ \forall a, b \in R$. Is S reflexive? Symmetric?

Solution:
1. reflexive: Is $aSa \ \forall a \in R$? Let $a \in R \Longrightarrow a - a = 0$
$\therefore S$ is not reflexive
2. symmetric? Let $(a, b) \in S$, $Is\ (b, a) \in S$?
$(a, b) \in S \Longrightarrow a - b > 0 \Longrightarrow b - a < 0 \Longrightarrow (b, a) \notin S$
$\therefore S$ is not symmetric

Example: (H. W.) Let $A = Z$ and define $a\, R\, b \Leftrightarrow |a| = |b| \ \forall a, b \in Z$
Is R reflexive? Symmetric?

Example: (H. W.) Let $A = Z$ and define $a\, R\, b \Leftrightarrow a = 1 \ \forall a \in Z$
$R = \{(1, b) : b \in Z\}$. Is R reflexive? Symmetric?

Theorem: A relation R on a set A is symmetric iff $R = R^{-1}$

Proof: $\Longrightarrow$**)** Suppose R is symmetric T. P. $R = R^{-1}$.
Let $(a, b) \in R \Leftrightarrow (b, a) \in R$ (R is symmetric)
$\Leftrightarrow (a, b) \in R^{-1}$(def. of R^{-1})

$\therefore R = R^{-1}$)

Proof: **$\Leftarrow$)** Suppose $R = R^{-1}$ T. P. R is symmetric

Let $(a,b) \in R \Longrightarrow (a,b) \in R^{-1}$ $(R = R^{-1})$

$\Longrightarrow (b,a) \in (R^{-1})^{-1} = R$

$\therefore R$ is symmetric.

Anti Symmetric Relation علاقة ضد التناظرية

A relation R on a set A is called anti symmetric if

$(a R b \wedge b Ra) \Longrightarrow a = b \,\forall a,b \in A$

R is not anti symmetric if $\exists a,b \in A$ s.t $(a Rb \wedge b Ra) \wedge a \neq b$.

Example: Let $A = \{1,2,3,4\}$ and R_1, R_2 are two relations on A such that $R_1 = \{(2,1),(3,1),(3,2),(1,1)\}$ and $R_2 = \{(2,1),(3,1),(1,2),(1,1)\}$. Are R_1, R_2 anti symmetric?

Solution:

R_1 is anti symmetric because ($1 R 1 \wedge 1 R1) \Longrightarrow 1 = 1$

R_2 is not anti symmetric because $\exists (2,1) \in R_2 \wedge (1,2) \in R_2$ but $1 \neq 2$

Example: (H. W.) Let $A = \{1,2,3,4\}$ and R is a relation on A such that $R = \{(4,2)\}$. Is R anti symmetric?

Example: (H. W.) Let $A = Z$ and R is a relation on Z such that $a R b \Leftrightarrow a = b + 1$ Is R reflexive? symmetric? Anti symmetric?

Hint : $R = \{(2,1),(1,0),(0,-1),\ldots\}$

R is not reflexive , R is not symmetric and R is anti symmetric

Example: Let $A = Z$ and R is a relation on Z such that $R b \Leftrightarrow a|b$. Is R anti symmetric?

Solution:

Let $a R b \wedge b R a \Longrightarrow a|b \wedge b|a$

$\Longrightarrow b = k_1 a \wedge a = k_2 b$, $k1, k2 \in Z$ .... $(*)$

$\Longrightarrow b = k_1(k_2 b) \Longrightarrow b = (k_1 k_2) b$

$\Longrightarrow k_1 k_2 = 1$

$\Longrightarrow k_1 = k_2 = 1$ or $k_1 = k_2 = -1$

If $k_1 = k_2 = 1$ $then$ $b = 1.a$ (from *)

If $k_1 = k_2 = -1$ $then\ b = -1.\ a$ (from *)
$\Rightarrow b = a\ or\ b = -a$
$\therefore R$ is not anti symmetric

Example: (H. W.) Let $A = Z$ and R is a relation on Z such that $R\ b \Leftrightarrow a + b = 2k, k \in Z$. Is R anti symmetric?

Theorem:

Let R be a relation on A, then R is anti symmetric iff $R \cap R^{-1} \subseteq I_A$.

Proof: ⟹) Let R is anti symmetric T. P. $R \cap R^{-1} \subseteq I_A$
Let $(a,b) \in R \cap R^{-1} \Rightarrow (a,b) \in R \wedge (a,b) \in R^{-1}$ (def. of ∩)
$\Rightarrow (a,b) \in R \wedge (b,a) \in R$
$\Rightarrow a = b$ (R is anti symmetric)
$\Rightarrow (a,b) \in I_A$
$\therefore R \cap R^{-1} \subseteq I_A$

Proof: ⟸) Let $R \cap R^{-1} \subseteq I_A$ T. P. R is anti symmetric
Let $aRb \wedge b\,Ra \Rightarrow (a,b) \in R \wedge (b,a) \in R$
$\Rightarrow (a,b) \in R \wedge (a,b) \in R^{-1}$
$\Rightarrow (a,b) \in R \cap R - 1$ (def. of ∩)
$\Rightarrow (a,b) \in I_A \Rightarrow a = b$

<u>Transitive Relation</u> العلاقة المتعدية

A relation R on a set A is transitive If
$(a,b) \in R \wedge (b,c) \in R\ then\ (a,c) \in R\ \forall a,b,c \in A$
Or If $aR\ b \wedge b\ R\ c\ then\ a\ R\ c\ \forall a,b,c \in A$.

A relation R on a set A is not transitive if
$\exists a,b,c \in A\ such\ that\ (a,b) \in R \wedge (b,c) \in R \wedge (a,c) \notin R$

Example: Let $A = \{1,2,3,4\}$. Which of these relations are transitive?

$R_1 = \{(1,1),(1,2),(2,3),(1,3)\}$. R_1 is transitive on A
because $(1,2) \in R_1 \wedge (2,3) \in R_1 \Rightarrow (1,3) \in R_1$
And $(1,1) \in R_1 \wedge (1,2) \in R_1 \Rightarrow (1,2) \in R1$

$R_2 = \{(1,2),(2,3)\}$ R_2 is not transitive on A because
$(1,2) \in R_2 \wedge (2,3) \in R_2$ but $(1,3) \notin R_2$.

Example: Let $A = N$. Define a relation R on A s.t $R = \{(a,b): a \leq b\}$. Is R transitive on A?

Solution: Let $(a,b) \in R \wedge (b,c) \in R \Rightarrow a \leq b \wedge b \leq c$
$\Rightarrow a \leq c \Rightarrow (a,c) \in R$ $\therefore R$ is transitive on .

Example: (H. W.) Let $A = N$. Define $a\ R\ b \iff a|b$ Prove that R transitive on A.

Example: (H. W.) Let $A = Z$ and define $a\ R\ b \iff ab \geq 0\ \forall a,b \in Z$ Show that R is transitive.

Example: (H. W.) Let $A = R$ and define $aSb \iff a - b > 0\ \forall a,b \in R$. Is S transitive?

<u>Equivalence Relation</u> علاقة التكافؤ

A relation R on a set A is called equivalence relation if and only if R is reflexive, symmetric and transitive.

Example: Let $A = Z$ and R is a relation on Z s.t $aRb \iff a + b = 2k$, $k \in Z$ Show that R is equivalence relation?

Solution:

reflexive: T. P. $a\ R\ a\ \forall a \in Z$, let $a \in Z \Rightarrow a + a = 2a \Rightarrow (a,a) \in R$
Symmetric: Let $(a,b) \in R$ T. P. $(b,a) \in R$
$(a,b) \in R \Rightarrow a + b = 2k \Rightarrow b + a = 2k \Rightarrow (b,a) \in R$
$\therefore R$ is Symmetric
Transitive: Let $(a,b) \in R \wedge (b,c) \in R$ To Prove $(a,c) \in R$
$(a,b) \in R \Rightarrow a + b = 2k_1, \quad k_1 \in Z$ … … (1)
$(b,c) \in R \Rightarrow b + c = 2k_2, \quad k_2 \in Z$ … … (2)
By summing up equations (1) and (2)
$\Rightarrow a + b + b + c = 2k_1 + 2k_2$
$\Rightarrow a + 2b + c = 2(k_1 + k_2)$
$\Rightarrow a + c = 2(k_1 + k_2 - b)$

$\Longrightarrow a + c = 2k\ ,\ k = k_1 + k_2 - b \in Z$
$\Longrightarrow (a, c) \in R \Longrightarrow \therefore$ R is transitive

Thus R is equivalence relation

Example:(H.W.) Which relations from previous examples are equivalence relations?

Equivalence classes: صفوف التكافؤ

Let R be an equivalence relation on A. The set of all elements that are related to an element $a \in A$ is called an equivalence class of a. The equivalence class of a is denoted by $[a]$.

لتكن R علاقة تكافؤ معرفة على A وليكن a عنصرا في A. فان مجموعة كل العناصر التي ترتبط وفق العلاقة R مع a تسمى صف تكافؤ ال a ويرمز لها ب. [a]

$$[\boldsymbol{a}] = \{\boldsymbol{x} \in \boldsymbol{A}: \boldsymbol{x} \sim \boldsymbol{a}\}$$

$$\boldsymbol{x} \in [\boldsymbol{a}] \Leftrightarrow \boldsymbol{x} \sim \boldsymbol{a}$$

$$\boldsymbol{x} \notin [\boldsymbol{a}] \Leftrightarrow \boldsymbol{x} \nsim \boldsymbol{a}$$

Example: Let $A = \{1,2,3,4\}.R = \{(1,1), (2,2), (1,2), (2,1), (3,3), (4,4)\}$ be an equivalence relation on A. Find all equivalence classes on A.
Solution:

$[1] = \{x \in A: x \sim 1\} = \{1,2\}$

$[2] = \{x \in A: x \sim 2\} = \{1,2\}$

$[3] = \{x \in A: x \sim 3\} = \{3\}$

$[4] = \{x \in A: x \sim 4\} = \{4\}$

Example: Let $A = \{-1,1,0\}$ and $= \{(a, b) \in A \times A: \sqrt{a}\,3 = \sqrt{b}\,3\}$.
Show that R is an equivalence relation. Find all equivalence classes on A.
Solution:

$R = \{(-1,-1), (1,1), (0,0)\}$

$[-1] = \{x \in A: x \sim -1\} = \{-1\}$

$[1] = \{x \in A: x \sim 1\} = \{1\}$

$[0] = \{x \in A: x \sim 0\} = \{0\}$

Example: Let $A = Z$ and $R = \{(a,b) \in Z \times Z: a - b = 3k,\ k \in Z\}$. Show that R is an equivalence relation. Find all different equivalence classes on Z.

Solution: reflex.: Let $a \in Z$ T.P. $(a,a) \in R$

$a \in Z \Longrightarrow a - a = 3(0), k = 0 \in Z$

Symm.: Let $(a,b) \in R \Longrightarrow a - b = 3k, k \in Z \Longrightarrow b - a = -3k = 3(-k), -k \in Z \Longrightarrow (b,a) \in R$

Trans.: Let $(a,b) \in R \wedge (b,c) \in R$ To Prove $(a,c) \in R$
$(a,b) \in R \Longrightarrow a - b = 3k_1, k_1 \in Z$ ... ... (1)
$(b,c) \in R \Longrightarrow b - c = 3k_2, k_2 \in Z$ ... ... (2)
By summing up equations (1) and (2)
$a - b + b - c = 3k_1 + 3k_2$
$a - c = 3(k_1 + k_2)$
$a - c = 3s, \quad s = k_1 + k_2 \in Z$
$(a,c) \in R$
$\therefore R$ is reflexive, symm. and trans.
$\therefore R$ is equivalence relation on Z

To find all equivalence classes, we start with

$[0] = \{x \in Z: x \sim 0\} = \{x \in Z: x - 0 = 3k, k \in Z\}$
. $= \{x \in Z: x = 3k, k \in Z\} = \{0,3,-3,6,-6,\dots\}$

$[1] = \{x \in Z: x \sim 1\} = \{x \in Z: x - 1 = 3k, k \in Z\}$
. $= \{x \in Z: x = 3k + 1, k \in Z\} = \{1,4,-2,7,-5,\dots\}$

$[2] = \{x \in Z: x \sim 2\} = \{x \in Z: x - 2 = 3k, k \in Z\}$
. $= \{x \in Z: x = 3k + 2, k \in Z\} = \{2,5,-1,8,-4,\dots\}$

$[3] = \{x \in Z: x \sim 1\} = \{x \in Z: x - 3 = 3k, k \in Z\}$.
$= \{x \in Z: x = 3k + 3, k \in Z\} = \{3,6,0,9,-3,\dots\} = [0]$

$[4] = [1]$

$[5] = [2]$

Example: (H. W.) Let $A = Z$ and $R = \{(a,b) \in Z \times Z: a - b = 5k, k \in Z\}$. Show that R is an equivalence relation. Find all different equivalence classes on Z.

Example: Let $A = N$. Define a relation R on A such that $R = \{(a, b): a = b\}$. Show that R is an equivalence relation. Find all equivalence classes on N.

Solution: R is an equivalence relation **(H. W.)**

$[1] = \{x \in N: x \sim 1\} = \{x \in N: x = 1\} = \{1\}$

$[2] = \{x \in N: x \sim 2\} = \{x \in N: x = 2\} = \{2\}$

$[3] = \{x \in N: x \sim 3\} = \{x \in N: x = 3\} = \{3\}$ ……. etc

Theorem: Let R be an equivalence relation on A, then:

1. $[a] \neq \emptyset \; \forall a \in A$
2. $a \sim b$ *if and only if* $[a] = [b]$
3. $a \nsim b \Leftrightarrow [a] \cap [b] = \emptyset$
4. $[a] \neq [b] \Leftrightarrow [a] \cap [b] = \emptyset$
5. $a \in [b] \Leftrightarrow [a] = [b]$

Proof1: $[a] = \{x \in A: x \sim a\}$
Since R is an equivalence relation $\Longrightarrow R$ is reflexive
. $\Longrightarrow a \sim a \; \forall a \in A$
. $\Longrightarrow a \in [a]$ (def. of equi. classes)
. $\Longrightarrow [a] \neq \emptyset$

Proof 2: $\Longrightarrow$) Suppose $a \sim b$ T. P $[a] \subseteq [b] \wedge [b] \subseteq [a]$
let $x \in [a] \Longrightarrow x \sim a \wedge a \sim b$
. $\Longrightarrow x \sim b$ (R is trans.)
. $\Longrightarrow x \in [b]$
$\therefore [a] \subseteq [b]$
Similarly, prove that $[b] \subseteq [a]$ (H. W.)
($\Leftarrow$) Let $[a] = [b]$ T. P. $a \sim b$
since R is reflexive $\Longrightarrow a \sim a$
. $\Longrightarrow a \in [a] = [b]$ (from hypo.)
. $\Longrightarrow a \in [b]$
. $\Longrightarrow a \sim b$

Proof 3: $\Longrightarrow$**)** Suppose $a \nsim b$ T. P $[a] \cap [b] = \emptyset$
suppose $[a] \cap [b] \neq \emptyset$
$\exists x \in [a] \cap [b] \Longrightarrow x \in [a] \land x \in [b]$
$\Longrightarrow x \sim a \land x \sim b$ (def. of equi. classes)
$\Longrightarrow a \sim x \land x \sim b$ (R is symm.)
$\Longrightarrow a \sim b$ (R is trans.) contradiction
$\therefore [a] \cap [b] = \emptyset$

($\Leftarrow$) suppose $[a] \cap [b] = \emptyset$ T. P. $a \nsim b$
suppose $a \sim b \Longrightarrow [a] = [b]$ (from 2)
. $\Longrightarrow [a] \cap [b] \neq \emptyset$ contradiction
$\therefore a \nsim b$

Proof4: $\Longrightarrow$) suppose $[a] \neq [b]$ and $[a] \cap [b] \neq \emptyset$
. $\Longrightarrow a \sim b$ (from 3)
. $\Longrightarrow [a] = [b]$ تناقض مع الفرض

($\Leftarrow$) suppose $[a] \cap [b] = \emptyset$ T. P. $[a] \neq [b]$

suppose $[a] = [b] \Longrightarrow [a] \cap [b] \neq \emptyset$ تناقض

$\therefore [a] \neq [b]$

<u>Definition</u>: Partition of a Set تجزئة المجموعة

A collection of subsets $\{A_i : i \in I \subseteq N\}$ of A is called partition of A if

تجمع من المجموعات الجزئية الغير خالية $\{A_i : i \in I \subseteq N\}$ من المجموعة A تسمى تجزئة ل A اذا حققت الشروط التالية.

1. $A_i \neq \emptyset \ \forall i \in I$

2. $A_i \cap A_j = \emptyset \quad \forall i \neq j$

3. $\bigcup_{i \in I} A_i = A$

Example: Let $A = \{0,1,2,3,5,-2\}$. Find two partitions of A.

Solution:

First partition The collection $A_1 = \{0,1\}$, $A_2 = \{2,3,5\}$, $A_3 = \{-2\}$

Second Partition (H.W.)

Theorem: Let R be an equivalence relation on a nonempty set A. Then the set of all different equivalence classes forms a partition for A.

Proof:

Let $P = \{[a]: a \in A\}$ the set of all different equivalent classes of A.
T. P. P is a partition of A

(1) $[a] \neq \emptyset \ \forall a \in A$ (from Theorem (1))

(2) Let $[a], [b]$ are two different equivalence classes

T. P. $[a] \cap [b] = \emptyset$

Since $[a] \neq [b] \Longrightarrow [a] \cap [b] = \emptyset$ (from Theorem (3))

(3) T.P. $\bigcup_{a\in A} [a] = A$ T. P. $\bigcup_{a\in A} [a] \subseteq A \wedge A \subseteq \bigcup_{a\in A} [a]$
let $x \in \bigcup_{a\in A} [a] \Longrightarrow x \in [a]$ for some $a \in A$

$\Longrightarrow x \in A$ ($[a] \subseteq A$)
$\therefore \bigcup_{a\in A} [a] \subseteq A$ ... ... (1)
let $x \in A \Longrightarrow x \in [a]$ for some $a \in A$
$\Longrightarrow x \in \bigcup_{a\in A} [a]$ ($[a] \subseteq \bigcup_{a\in A} [a]$)
$\therefore A \subseteq \bigcup_{a\in A} [a]$ …….(2)
From (1) and (2), $\bigcup_{a\in A} [a] = A$

Example: Let $A = Z$ and define $a\,R\,b \Leftrightarrow |a| = |b| \ \forall a, b \in Z$ Prove that R is an equivalence relation. Then find all different equivalence classes (i.e., find a partition set of Z).

Solution: R is an equivalence relation (H. W.)

$[0] = \{x \in Z: x \sim 0\} = \{0\}$

$[1] = \{1, -1\}$

$[2] = \{2, -2\}$

.

.

.. etc

....

Partition set: $P = \{[a]: a \in Z^{+}\}$

Example: Let $A = Z$ and $R = \{(a, b) \in Z \times Z: a - b = 3k, k \in Z\}$. Find the partition set on Z.

Solution: From Example, the relation R is an equivalence relation
Partition set=The set of all different equivalence classes
$P = \{[0], [1], [2]\}$

....

Order Relations: **علاقات الترتيب**

1. Partially Ordered Relation: العلاقات المرتبة جزئيا

A relation R on a set A is called Partially Ordered Relation (P. O. R) or partially ordering if it is reflexive, anti-symmetric and transitive. The pair (A, R) is called partially ordered set (P.O.S) .

العلاقة على المجموعة A تسمى مرتبة جزئيا اذا كانت العلاقة انعكاسية وضد تناظرية ومتعدية كما يسمى الزوج (A, R) بالمجموعة المرتبة جزئيا.

Mathematically,

R is P.O.R $\Leftrightarrow$ R reflexive $\wedge$ anti symmetric $\wedge$ transitive

R is not P.O.R $\Leftrightarrow$ R not reflexive $\vee$ not anti- symmetric $\vee$ not transitive.

Example: (H.W.) Let $A = \{1,2,-3\}$ be a set. Let

$R_1 = \{(a,b)) \in A \times A: a \geq b\}$
$= \{(1,1),(2,2),(-3,-3),(1,-3),(2,1),(2,-3)\}$

$R_2 = \{(1,1),(2,2),(-3,-3),(1,2),(2,1)\}$

$R_3 = \{(1,1),(2,2),(-3,-3)\} = I_A$

Are (A, R_1), (A, R_2) and (A, R_3) partially ordered sets?

Example: Show that $(Z, \geq)$ is a partially ordered set

Solution: Let R be a relation such that $R = \{(a,b) \in Z \times Z: a \geq b\}$

We must show R is reflexive, anti-symmetric and transitive

Reflexive: since $a \geq a \Longrightarrow (a,a) \in R \Longrightarrow R$ is reflexive

Anti Symmetric: let $(a,b) \in R \wedge (b,a) \in R$ T. P. $a = b$

$$\Longrightarrow a \geq b \Longrightarrow a = r + b,\ r \geq 0 \ \dots . (1)$$

And $$b \geq a \Longrightarrow b = s + a,\ s \geq 0 \ \dots . (2)$$

Substitute (1) in (2) $\Longrightarrow b = s + r + b$

$$\Longrightarrow s + r = 0 \quad but\ r, s \geq 0 \Rightarrow r = s = 0$$

....

Substitute $r = 0$ in (1) $\Longrightarrow a = b$ then R is anti-symmetric

Transitive: let $(a,b) \in R \wedge (b,c) \in R \Longrightarrow a \geq b \wedge b \geq c$

$$\Longrightarrow a \geq b \Longrightarrow a = r + b, r \geq 0 \dots .(1)$$

And $$b \geq c \Longrightarrow b = s + c, s \geq 0 \dots .(2)$$

Substitute (2) in (1) $\Longrightarrow a = (r + s) + c \; and \; r + s \geq 0$

$\Longrightarrow a \geq c \Longrightarrow (a,c) \in R \Longrightarrow$ R is transitive.

Remark: For any nonempty set A, the relation $A \times A$ is not $P.O.R$

Example: Let $A = Z$ and $R = Z \times Z$. Show that R is not $P.O.R$ The relation R is an equivalence relation $\Longrightarrow R$ is symmetric

$\Longrightarrow (a,b) \in R$ and $(b,a) \in R \; \forall a,b \in R$

But $a \neq b$ (in general) $\Longrightarrow R$ is not anti symmetric

$\Longrightarrow R$ is not $P.O.R$

Example: (H .W.) Let $A = Z$ and $R_1 = \{(a,b) \in Z \times Z : a|b\}$

$R_2 = \{(a,b) \in Z \times Z : a \leq b\}$

$R_3 = \{(a,b) \in Z \times Z : a < b\}$

$R_4 = \{(a,b) \in Z \times Z : a > b\}$

Show that $(Z, a|b)$ is not a partially ordered set

Show that R_2 is $P.O.R$

Show that R_3 and R_4 are not $P.O.R$

Example: (H .W.) Let $X = \{1,2,3\}$

and $R = \{(A,B) \in P(X) \times P(X) : A \subseteq B\}$

Show that R is a partially ordered relation on $P(X)$

Definition: Let R be a $P.O.R$ on a set A and let $a,b \in A$. Then a,b are called **comparable** عنصرين قابلين للمقارنة with respect to R if $(a,b) \in R$ or $(b,a) \in R$.

Mathematically,

a, b are comparable $\Leftrightarrow (a,b) \in R \vee (b,a) \in R$

a, b are not comparable $\Leftrightarrow (a,b) \notin R \wedge (b,a) \notin R$

Example:

Let $A = \{1,2,3\}$, let $R = \{(1,1),(2,2),(3,3),(1,2),(3,1),(3,2)\}$ be a $P.O.R$ Find the comparable element in A with respect to R

$1R\,1 \Longrightarrow 1,1$ are comparable

$2R2 \Longrightarrow 2,2$ are comparable

$3R3 \Longrightarrow 3,3$ are comparable

$1R2 \Longrightarrow 1,2$ are comparable

$3R1 \Longrightarrow 1,3$ are comparable

$2R3 \Longrightarrow 2,3$ are comparable

Example: **(H. W.)** Let $A = \{3,4,6,8,10\}$ and $R = \{(a,b) \in A \times A: a\,|b\}$ Find the comparable element in A with respect to R.

2. Totally Ordered Relation العلاقة المرتبة كليا

A relation R on a set A is called totally ordered relation $(T.O.R)$ or totally ordering if

1. $R\ is\ P.O.R$

2. a, b are comparable $\forall a, b \in A$

العلاقة على المجموعة A تسمى مرتبة كليا اذا كانت العلاقة مرتبة جزئيا وكل عنصرين في A قابلين للمقارنة بالنسبة للعالقة

Example:

Let $A = \{1,2,-3\}$ be a set.

Let $R = \{(a,b)) \in A \times A: a \geq b\}$
$= \{(1,1),(2,2),(-3,-3),(1,-3),(2,1),(2,-3)\}$

Is $R\ T.O.R$?

Solution:

1. $P.O.R$: From Example, R is $P.O.R$

2.Comparable: From reflexive relation, each element is comparable with itself

$(1,-3) \in R \Longrightarrow 1,-3$ are comparable

$(2,1) \in R \Longrightarrow$ 1,2 are comparable

$(2,-3) \in R \Longrightarrow 2,-3$ are comparable

$\therefore a, b$ are comparable $\forall a, b \in A$

$\therefore R$ is $T.O.R$

Example: Show that $(Z, \geq)$ is a totally ordered set

Solution:

$P.O.R$: From Example, R is $P.O.R$

Comparable: T.P. $a \geq b \vee b \geq a\ \forall a, b \in A$

Let $a, b \in Z \Longrightarrow a = b \vee a > b \vee b > a$

$\Longrightarrow a \geq b \vee a \geq b \vee b \geq a$

$\Longrightarrow a \geq b \vee b \geq a$

$\Longrightarrow aRb \vee bRa$

$\therefore a, b$ are comparable $\forall a, b \in Z$

$\therefore R$ is $T.O.R$

Example: **(H .W.)** Show that $(Z, \leq)$ is a totally ordered set .

Example: **(H .W.)** $(R, \leq)$, $(R, \geq)$, $(Q, \geq)$, $(Q, \leq)$, $(N, \leq)$, $(N, \geq)$ are totally ordered sets

Example: **(H .W.)** Give an example of a $P.O.R$ that is not $T.O.R$.