Chapter Two

1- Estimation Theory

Let $x_1, x_2, ..., x_n$ be a r.s. from a distribution having P.d.f $f(x, \theta)$, where $f(x, \theta)$ is of known from with unknown parameter θ, therefore it have to be estimator from the sample date. Two types of estimation can be done, namely the point estimation and the interval estimation.

Def:

The point estimation of θ is a rule (function) that assigns each element of the sample a value (estimate) of θ denoted as $\hat{\theta} = (x_1, x_2, ..., x_n)$.

Properties of Good Estimator

(1) Unbiasedness:

An estimator $\hat{\theta}$ is said to be unbiased estimator of θ if $\boxed{E(\hat{\theta}) = \theta}$. Otherwise, the estimator is said to be biased.

The value of biased $b(\theta)$ is defined as

$$\boxed{b(\theta) = E(\hat{\theta} - \theta) = \; = E(\hat{\theta}) - \theta}$$

Ex:

Let $x_1, x_2, ..., x_n$ be a r.s from $N(\mu, 1)$ show that $\hat{\theta} = \bar{x}$ is unbiased estimator of μ.

Solution:

We have to show that $E(\bar{X}) = \mu$

$$E(\overline{X}) = E(\frac{\sum x_i}{n}) = \frac{1}{n} E(\sum x_i) = \frac{1}{n} E(x_i)$$

since $x_i \sim N(\mu, 1)$, then $E(x_i) = \mu$

$$E(\overline{x}) = \frac{1}{n} \sum_{i=1}^{n} \mu = \frac{1}{n} n\mu = \mu$$

$\hat{\theta} = \overline{x}$ is unbiased estimator of μ

Ex:

Let $x_1, x_2, ..., x_n$ be ar.s from $N(\mu, \delta^2)$, show that $\frac{1}{n-1} \sum (x_i - \overline{x})^2$ is unbiased est. of δ^2

Solution:

Recalling that $S^2 = \frac{1}{n} \sum (x_i - \overline{x})^2$

$$\sum (x_i - \overline{x})^2 = nS^2 \Rightarrow E(\frac{1}{n-1} \sum (x_i - \overline{x})^2) = \frac{1}{n-1} E(nS^2)$$

$$\frac{n}{n-1} E(S^2) = \frac{n}{n-1} \left[\frac{n-1}{n} \delta^2 \right] = \delta^2$$

$\Rightarrow \frac{1}{n-1} \sum (x_i - \overline{x})^2$ is unbiased est. of δ^2.

(2) Mean Square Error متوسط مربعات الخطأ

The mean square error (MSE) of an est. $\hat{\theta}$ is defined as

$$\boxed{MSE\ (\hat{\theta}) = E\ (\hat{\theta} - \theta)^2 = var\ (\hat{\theta}) + b^2(\theta)}$$

if $\hat{\theta}$ is unbiased then $b(\theta) = 0$ and MSE ($\hat{\theta}$) = var ($\hat{\theta}$).

The good estimator has MSE as small as possible.

Ex.

Let $x_1, x_2, ..., x_n$ be ar.s from $f(x, \theta) = \theta^x (1-\theta)^{1-x}$, $x = 0, 1$, use MSE to compare between the two statistics (estimators) $\bar{x}, x_i$.

Solution:

Since $x_i \sim$ Bernoulli $(1, \theta)$, then $E(x_i) = \theta$, $i = 1, 2, ..., n$, and hence $\varepsilon(x_i) = \theta$

Also, $E(\bar{x}) = E(\frac{\sum x_i}{n}) = (\frac{1}{n})E(\sum x_i) = \frac{1}{n}\sum E(x_i) = \frac{1}{n}\sum \theta$

$= \frac{1}{n} n\theta = \theta$

Each of $\bar{x}, x_i$ are unbiased est. of θ.

MSE (x_i) = var $(x_i) = \theta(1-\theta)$

MSE$(\bar{x})$ = var $(\bar{x})$ = $\text{var}(\frac{\sum x_i}{n}) = \frac{1}{n^2}\,\text{var}\sum x_i = \frac{1}{n^2}\sum \text{var}(x_1)$

$= \frac{1}{n^2}\sum \theta(1-\theta) = \frac{1}{n^2} n\theta(1-\theta) = \frac{\theta(1-\theta)}{n}$

$\text{MSE}(\bar{x}) < \text{MSE}(x_i)$. This means that $\bar{x}$ is better than x_i.

(3) Consistency الاتساق

$\hat{\theta}$ is consistent est. of θ if

1) $\hat{\theta}$ is unbiased
2) $\lim_{n\to\infty} \text{var}(\hat{\theta}) = 0$

Ex.

Let $x_1, x_2, ..., x_n$ be ar.s from p(θ). Show that $\hat{\theta} = \overline{x}$ is consistent est. of θ.

Solution:

Since x ~ p(θ), then $f(x, \theta) = \frac{e^{-\theta}\theta^x}{x!}$, x = 0, 1, 2, ...

$$E(\hat{\theta}) = E(\overline{x}) = E(\frac{\sum x_i}{n}) = \frac{1}{n}\sum E(x_i) = \frac{1}{n}\sum \theta = \frac{1}{n} n\theta$$

$= \theta \Rightarrow \hat{\theta} = \overline{x}$ is unbiased est. of θ.

$$\text{var}(\hat{\theta}) = \text{var}(\frac{\sum x_i}{n}) = \frac{1}{n^2}\sum \text{var}(x_i) = \frac{1}{n^2}\sum \theta = \frac{n\theta}{n^2} = \frac{\theta}{n}$$

$\lim_{n\to\infty} \text{var}(\hat{\theta}) = \lim_{n\to\infty} \frac{\theta}{n} = 0$, the two conditions consistency are satisfied $\Rightarrow \hat{\theta} = \overline{x}$ is consistent est. of θ

(4) Minimum Variance Unbiased Estimate

If a statistic $T = t(x_1, x_2, ..., x_n)$ is such that

1) T is unbiased statistic of θ.

2) It has smallest variance among all the unbiased statistics of θ, then T is called a minimum variance unbiased estimate (MVUE) of θ.

Ex.

Let y_1 and y_2 be two stochastically independent unbiased statistics for θ. Say the variance of y_1 is twice the variance of y_2. Find the constants k_1 and k_2 so that $k_1y_1 + k_2y_2$ is an unbiased statistic with smallest possible variance for such a linear combination.

Solution:

Since each of y_1, y_2 and $k_1y_1 + k_2y_2$ are unbiased then $E(y_1) = \theta$, $E(y_2) = \theta$, $E(k_1y_1 + k_2y_2) = k_1E(y_1) + k_2E(y_2) = \theta$

$k_1\theta + k_2\theta = \theta \Rightarrow (k_1 + k_2)\theta = \theta$

$k_1 + k_2 = 1 \Rightarrow \boxed{k_2 = 1 - k_1}$

let $\delta^2 = var(y_2) \Rightarrow var(y_1) = 2\delta^2$

Putting $Q = var(k_1y_1 + k_2y_2)$ then

$$Q = k_1^2 \, var(y_1) + k_2^2 \, var(y_2)$$

$$= 2k_1^2\delta^2 + (1-k_1)^2\delta^2$$

$$\frac{\partial Q}{\partial k_1} = 4k_1\delta^2 - 2(1-k_1)\delta^2 = 0$$

$$4k_1 - 2 + 2k_1 = 0 \Rightarrow 6k_1 = 2 \Rightarrow k_1 = \frac{1}{3}$$

$$k_2 = 1 - k_1 = 1 - \frac{1}{3} = \frac{2}{3}$$

(5) Efficiency الكفاءة

Let T be unbiased est. for a parameter θ. Then T is called an efficient estimator of θ iff the variance of T attains the Rao-Cramer lower bound given by:

$$\boxed{var(T) \geq \frac{1}{nE(\frac{\partial \ln f(x,\theta)}{\partial\theta})^2}}$$

it can be shown that:

$$\boxed{E(\frac{\partial \ln f(x,\theta)}{\partial\theta})^2 = -E(\frac{\partial^2 \ln f(x,\theta)}{\partial\theta^2})}$$

Ex.