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# CHAPTER Two

## Set Theory نظرية المجموعات

## Contents:

1. Basic notion of sets مفهوم المجموعات
2. Subsets المجموعات الجزئية
3. Algebra of sets (union, intersection, difference, complement, symmetric difference جبر المجموعات أو العمليات على المجموعات

**Definition : Set المجموعة**

A set is an unordered collection of objects. The objects are called the elements or members of the set.

المجموعة هي تجمع من الاشياء المعرفة بدون ترتيب و التي تسمى بالعناصر أو أعضاء تنتمي للمجموعة .

**Remarks:**

1. The capital letters usually used to represents sets such as A, B, C,…etc.

2. The small letters such as a, b, c, d,…etc are used to represents the members or the elements of the set.

3. Membership in a set is denoted as follows:

| |
|---|
| $\boldsymbol{a} \in \boldsymbol{A}$ denotes that $\boldsymbol{a}$ belongs to a set $A$ |

4. Non-membership to a set is denoted as follows:

| |
|---|
| $\boldsymbol{a} \notin \boldsymbol{A}$ denotes that $\boldsymbol{a}$ does belong to a set $A$ |

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## Specifying a Set: طرق التعبير عن المجموعة

**1. Listing members of a set:** الطريقة الجدولية

In this way, we list all non-repeated members of a set separated by commas and contained in braces { }. The members are not in an order.

الطريقة الجدولية او طريقة القائمة : في هذه الطريقة نضع جميع العناصر الغير المعادة بين قوسي مجموعة وبفواصل تفصل بينها. عناصر المجموعة لا يشترط أن تكون مرتبة بطريقة معينة.

**Example:**

1. $A = \{1,2,-5,0,9\}$, $B = \{x, y, Ali, fish\}$, $C = \{z_1 \ , z_2 \ , z_3\}$ are sets
2. The set of vowel letters in English: $V = \{a, e, i, o, u\}$
3. The set of even positive numbers less than 8 is: $W = \{0,2,4,6\}$.
4. The set of positive numbers less than 50 is: $K = \{1,2, \dots ,49\}$

**2. Listing a set property**: استخدام الصفة المميزة للمجموعة

In this way, we state the property that characterize the elements in a set in as follows: $\{x: p(x)\}$, where $x$ is a variable and $p(x)$ is an open sentence.

**Example**:

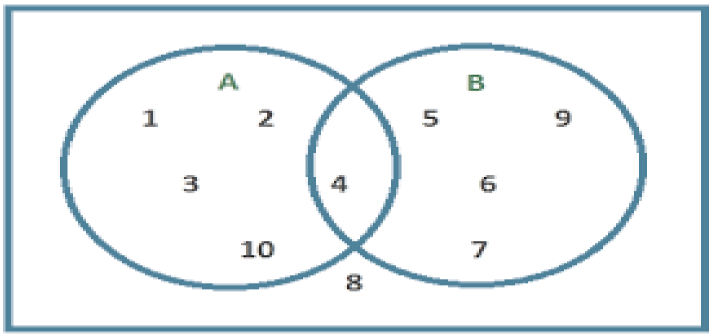
$A = \{x: x \in Q\}$

$B = \{x: x \text{ } is \text{ } poistive \text{ } odd \text{ } and \text{ x} < 10\}$ = {1,3,5,7,9}

$C = \{x \in N: -3 \leq x \leq 5\}$ = {1,2,3,4,5}

**3. Venn Diagrams**: مخططات فن

في هذه الطريقة توضع عناصر المجموعة داخل منحني مغلق يمثل المجموعة وتستخدم هذه الطريقة لأغراض توضيحية فقط .




**Empty Set** المجموعة الخالية

The set that contains no elements is called an empty set and is denoted by

{ } or ∅.

**<u>Example:</u>**

$A = \{x \in N: 2 < x < 3\} = \emptyset$

$B = \{x \in E: \sqrt{x} = 1\} = \emptyset$

$C = \{x \in N: x < 0\} = \{\,\}$

**<u>Subsets:</u>** المجموعات الجزئية

The set $A$ is a subset of a set $B$ $(A \subseteq B)$ if and only if every element of $A$ is an element of $B$. In other words,

$$A \subseteq B \; iff \; \forall x, x \in A \Rightarrow x \in B$$

**<u>Remark:</u>** A is not a subset of B is denoted by A$\nsubseteq B$.

$A \nsubseteq B$ if and only if $\sim[\forall x, x \in A \Rightarrow x \in B]$

if and only if $\exists x; x \in A \wedge x \notin B$

**<u>Example:</u>** Consider the sets $A = \{2\}$, $B = \{1, 2, 3\}$ and $C = \{4, 5\}$ and $D = \{-2, 1, 2, 3, 4, 5\}$. Then $A \subseteq$ B, . A$\subseteq D$, B $\subseteq D$ and $C \subseteq D$. It is true that A $\subseteq$ A , $B \subseteq B$, $C \subseteq C$ and $D \subseteq D$.

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**Theorem:** Let $A, B$ and $C$ be any sets, then

1.$\emptyset \subseteq A$

2. $A \subseteq A$

3. If $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$

**Proof 1:** T. P $\emptyset \subseteq A$, i.e., T. P $\forall x \in \emptyset \Rightarrow x \in A$

$F \Rightarrow (T \; or \; F) = T$

$\Rightarrow \emptyset \subseteq A$.

**Proof 2:** T. P $A \subseteq A$, i.e., T. P $\forall x \in A \Rightarrow x \in A$

$T \Rightarrow T = T$

$\Rightarrow A \subseteq A$.

**Proof 3:** T. P If $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$

T. P $\forall x, x \in A \Rightarrow x \in C$

Since $A \subseteq B \Rightarrow \forall x, x \in A \Rightarrow x \in B$

And $B \subseteq C \Rightarrow \forall x, x \in B \Rightarrow x \in C$

Then $\forall x, x \in A \Rightarrow x \in B \Rightarrow x \in C$

$\Rightarrow \forall x, x \in A \Rightarrow x \in C \Rightarrow A \subseteq C$

**Proper Subset** المجموعة الجزئية الفعلية

A set $A$ is called a proper subset of $B$ and denoted by ( $A \subset B$) if and only if $A \subseteq B$ and there exist an element $x \in B$ that is $x \notin A$.

i.e., $A \subseteq B$ iff $\{\forall x, x \in A \Rightarrow x \in B\} \wedge \{\exists y, y \in B \wedge y \notin A\}$

**Example:** Consider the sets $A = \{2\}$, $B = \{1, 2, 3\}$ , $A \subset B$

since $\exists$ 1 $\in B$ but 1 $\notin A$.

**Equal Sets** **المجموعات المتساوية**

Two sets $A$ and $B$ are equal if they both have the same elements or, equivalently, if each is contained in the other.

$\boldsymbol{A = B \ iff \ A \subseteq B \wedge B \subseteq A}$

$\leftrightarrow \{\forall \boldsymbol{x}, \boldsymbol{x} \in \boldsymbol{A} \rightarrow \boldsymbol{x} \in \boldsymbol{B}\} \wedge \{\forall \boldsymbol{x}, \boldsymbol{x} \in \boldsymbol{B} \rightarrow \boldsymbol{x} \in \boldsymbol{A}\}$

$\leftrightarrow$ **{ ∀x,x** $\in B \leftrightarrow$ **x** $\in C$**}.**

**Lemma:** ( H. W. ) Prove that: $A = A$, for any set $A$.

**Definition: Universal Set** **المجموعة الشاملة**

Universal set $V$ is the set that contains all the elements or the sets we have under discussion.

المجموعة الشاملة: هي المجموعة التي تحوي جميع العناصر أو المجموعات قيد المناقشة ويرمز لها بالرمز U .

**Example:** Let $A = \{x, y, 3\}$, $B = \{2, -5, 100\}$, $C = \{2,3,1\}$

Find a universal set U.

**Example:** Let $A = \{x \in R : 2 \leq x \leq 5\}$ and $B = \{x \in R : -1 \leq x \leq 2\}$

Find a universal set U.

**Definition: Family of Sets** عائلة المجموعات

Family of sets is a set that have other sets as members.

يقال للمجموعة التي يكون كل عنصر من عناصرها مجموعة أنها عائلة مجموعات

Example2.25:

1. $A = \{\{1\},\{2\}\}$ is a family of sets

2. $B = \{\emptyset\}$ is a family of a set

3. $X = \{X\}$ is a family of a set

4. $A = \{\{x\},\{y, z\},\{1, \dots, 5\}\}$

5. $H = \{A: A \ is \ a \ subset \ of \ \{1,2,3\}\}$

6. $K = \{A_i : A_i = \{2, \frac{2}{i}\}, i = 1,2,3\}$

**Definition:** **Power Set** مجموعة القوى أو مجموعة الأجزاء

Given a set $X$, the power set of $X$ is the set of all subsets of $X$. The power set of $X$ is denoted by $P(X)$.

لتكن X مجموعة يقال لمجموعة كل المجموعات الجزئية من X أنها مجموعة القوى ل X ويرمزلها بالرمز. P(X) .

$$P(X) = \{A: A \subseteq X\}, \qquad A \in P(X) \Leftrightarrow A \subseteq X$$

**Example:** Find $P(X)$ for the following sets X:

1. X={1,2,3} , $P(X)$ ={X, ∅, {1}, {2},{3},{1,2},{1,3},{2,3}}

2. $X = \{1,2,a\}$, $P(X) = \{\emptyset, X, \{1\}, \{2\}, \{a\}, \{1,2\}, \{1,a\}, \{2,a\}\}$

3. $X = \{\emptyset\}$, $P(X) = \{\emptyset, X\}$

4. $X = \{\{-2\},3\}$, $P(X) = \{\emptyset, X, \{-2\}, \{3\}\}$

**Remark:**

1. Since $X \subseteq X$, then P(X) ≠ ∅.

2. If $X$ is finite and has $n$ elements, then $P(X)$ has $\mathbf{2^n}$ elements.

**Theorem:** Let $X, Y$ be any sets, then $\boldsymbol{X \subseteq Y \Leftrightarrow P(X) \subseteq P(Y)}$

**Proof: (⟹)** Let $X \subseteq Y$ T.P $P(X) \subseteq P(Y)$

Let $A \in P(X) \Rightarrow A \subseteq X$ (By def. of $P(X)$)

$\Rightarrow A \subseteq Y$ $(X \subseteq Y)$

$\Rightarrow A \in P(Y)$

$\therefore P(X) \subseteq P(Y)$

**Proof: (⟸)** Let $P(X) \subseteq P(Y)$ To Prove $X \subseteq Y$

Let $x \in X \Rightarrow \{x\} \subseteq X$

$\Rightarrow \{x\} \in P(X)$

$\Rightarrow \{x\} \in$ P(Y) $\{ P(X) \subseteq P(Y)\}$

$\Longrightarrow \{x\} \subseteq Y$

$\Longrightarrow x \in Y$

$\therefore X \subseteq Y$

# Algebra of sets:

## 1. Union الاتحاد

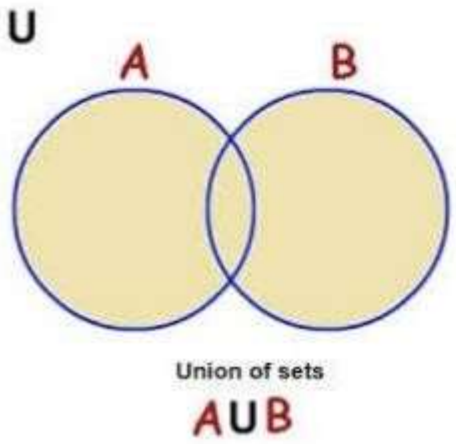
The union of the sets $A$ and $B$, denoted by $A \cup B$, is the set of elements which belong to A or to B.

اتحاد مجموعتين A و B : هي مجموعة العناصر التي تنتمي للمجموعة A أو المجموعة B.

$\boldsymbol{A \cup B = \{x, x \in A \vee x \in B\}}$

$\boldsymbol{x \in A \cup B \Leftrightarrow x \in A \vee x \in B}$

$\boldsymbol{x \notin A \cup B \Leftrightarrow x \notin A \wedge x \notin B}$




**Example:** Let $A = \{x \in N : 1 \leq x \leq 5\} = \{1,2,3,4,5\}$

$B = \{x \in N: 8 \leq x \leq 12\} = \{8,9,10,11,12\}$

Find $A \cup B$, $B \cup A$, $A \cup A$, and $B \cup \emptyset$

**Solution**: $A \cup B = B \cup A = \{1, \dots, 5, 8, \dots, 12\}$

$A \cup A = A$, $\quad B \cup \emptyset = B$

**Example:**(H.W.) Let $A = \{x \in R: -2 \leq x \leq 5\} = [-2,5]$,

$B = \{x \in E: y^2 - 16 = 0\} = \{4, -4\}$

$C = \{1,4\}$

Find $A \cup (B \cup C)$, $(A \cup B) \cup C$, $P(B), P(C)$, $P(B \cup C)$

**Definition:** **Generalization of the union** تعميم الاتحاد

Let $A_1, A_2, \ldots, A_n, A_{n+1}$ ... be any sets. Then:

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n$$

اتحاد عدد منتهي من المجموعات

$$\bigcup_{i=1}^{\infty} A_i = A_1 \cup A_2 \cup \ldots \cup A_n \cup A_{n+1} \ldots$$

اتحاد عدد غير منتهي من المجموعات

**Example:** Let $H = \{A_i \,;\, A_i = \{2i + 3\}, i \in Z\}$

Find $\bigcup_{i=1}^{4} A_i$ , $\bigcup_{i=-1}^{3} A_i$

**Solution:**

$\bigcup_{i=1}^{4} A_i = A_1 \cup A_2 \cup A_3 \cup A_4$

$= \{5\} \cup \{7\} \cup \{9\} \cup \{11\} = \{5,7,9,11\}$

$\bigcup_{i=-1}^{3} A_i = A_{-1} \cup A_0 \cup A_1 \cup A_2 \cup A_3$

$= \{1\} \cup \{3\} \cup \{5\} \cup \{7\} \cup \{9\} = \{1,3,5,7,9\}$

**Example**: (H. W.) Let $K = \{A_n \,;\, A_n = (-n, n), n \in N\}$

Find $A_1 \cup \bigcup_{n=2}^{\infty} A_n$

**Example:** (H. W.) Let $K = \{A_n \,;\, A_n = [n, n + 10), n \in Z\}$

Find $\bigcup_{n=-2}^{2} A_n$

**Example:** (H. W.) Let K = $\{A_j \,;\, A_j = \{j + 1, j + 2\}, j \in N\}$

Find $\bigcup_{j=3}^{10} A_n$

**Theorem:** Let $A, B$ and $C$ any three sets. Then:

1. $A \cup \emptyset = A$ (Identity law)

2. $A \cup A = A$ (Idempotent law)

3. $A \cup U = U$ (Domination law)

4. $(A \cup B) \cup C = A \cup (B \cup C)$ (H.W.)

5. $A \cup B = B \cup A$

6. $A \subseteq B \Leftrightarrow A \cup B = B$

7. $A \subseteq A \cup B, B \subseteq A \cup B$(H.W.)

8. P(A) ∪ P(B) ⊆ P(A ∪ B)

**Proof 1:** To prove $A \cup \emptyset \subseteq A \wedge A \subseteq A \cup \emptyset$

T. P $A \cup \emptyset \subseteq A$ (T. P $\forall x \in A \cup \emptyset \Longrightarrow x \in A$)

Let $x \in A \cup \emptyset \Longrightarrow x \in A \vee x \in \emptyset$ (def. of ∪)

$\Longrightarrow x \in A \vee F$

$\Longrightarrow x \in A$ $(p \vee F = p)$

$\therefore A \cup \emptyset \subseteq A$ ……………(1)

Let x ∈ A⟹ x ∈ A ∨ F (p ∨ F=p)

⟹ x ∈ A ∨ $x \in \emptyset$

$\Longrightarrow x \in A \cup \emptyset$ (def. of ∪)

$\therefore A \subseteq A \cup \emptyset$ ……………(2)

From (1) & (2), $A \cup \emptyset = A$

**Proof 2:** To prove $A \cup A \subseteq A \wedge A \subseteq A \cup A$

T. P $A \cup A \subseteq A$ (T. P $\forall x \in A \cup A \Longrightarrow x \in A$)

Let $x \in A \cup A \Longrightarrow x \in A \lor x \in A$ (def. of ∪)

$\Longrightarrow x \in A$ (p ∨ p=p)

$\therefore A \cup A \subseteq A$ ……………(1)

Let $x \in A \Longrightarrow x \in A \lor x \in A$ (p ∨ p=p)

$\Longrightarrow x \in A \cup A$ (def. of ∪)

$\therefore A \subseteq A \cup A$ ……………(2)

From (1) & (2), $A \cup A = A$

**Proof 3:** To prove $A \cup U \subseteq U \land U \subseteq A \cup U$

T. P $A \cup U \subseteq U$ (T. P $\forall x \in A \cup U \Longrightarrow x \in U$)

Let $x \in A \cup U \Longrightarrow x \in A \lor x \in U$ (def. of ∪)

$\Longrightarrow x \in U \lor x \in U$ ($A \subseteq U$)

$\Longrightarrow x \in U$

$\therefore A \cup U \subseteq U$ ……………(1)

Let $x \in U \Longrightarrow x \in U \lor x \in A$ (T ∨ p=T)

$\Longrightarrow x \in A \cup U$ (def. of ∪)

$\therefore U \subseteq A \cup U$ ……………(2)

From (1) & (2), $A \cup U = U$

**Proof 5:** To prove $A \cup B \subseteq B \cup A \land B \cup A \subseteq A \cup B$

T.P $A \cup B \subseteq B \cup A$

Let x∈ $A \cup B \Longrightarrow x \in A \lor x \in B$ (def. of ∪)

$\Longrightarrow x \in B \lor x \in A$ (∨ is commutative)

$x \in B \cup A$ (def. of ∪)

$\therefore A \cup B \subseteq B \cup A$ ……………(1)

Similarly, show that $B \cup A \subseteq A \cup B$ (H. W.) .................... (2)

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From (1) & (2), $A \cup B = B \cup A$

**Proof 6:** To prove $A \subseteq B \Leftrightarrow A \cup B = B$

**(⟹)** if $A \subseteq B$ T. P $A \cup B = B$

T. P $A \cup B \subseteq B \wedge B \subseteq A \cup B$

Let $x \in A \cup B \Longrightarrow x \in A \vee x \in B$ (def. of ∪)

$\Longrightarrow x \in B \vee x \in B$ $(A \subseteq B)$

$\Longrightarrow x \in B$ $(P \vee P = P)$

$\therefore A \cup B \subseteq B$……………(1)

Similarly, show that $B \subseteq A \cup B$ (H. W.) .................... (2)

From (1) & (2), $A \cup B = B$

(⟸) if $A \cup B = B$ T. P $A \subseteq B$

Let $x \in A$

$\Longrightarrow x \in A \cup B$ (def. of ∪)

$\Longrightarrow x \in B$ $(A \cup B = B)$

$\therefore A \subseteq B$

**Proof 8:** T. P $P(A) \cup P(B) \subseteq P(A \cup B)$

Let $X \in P(A) \cup P(B)$ To prove $X \in P(A \cup B)$

$X \in P(A) \cup P(B) \Longrightarrow X \in P(A) \vee X \in P(B)$ (def. of ∪)

$\Longrightarrow X \subseteq A \vee X \subseteq B$ (def. of P(A))

$\Longrightarrow X \subseteq (A \cup B)$ (def. of ∪)

$\Longrightarrow X \in P(A \cup B)$ (def. of $P(A \cup B)$)

$\therefore P(A) \cup P(B) \subseteq P(A \cup B)$

## 2. Intersection التقاطع

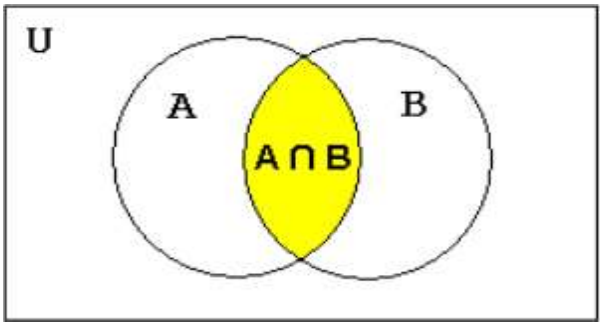
The intersection of the sets A and B, denoted by A ∩ B, is the set of elements which belong to both A and to B.

تقاطع مجموعتين B و A هي مجموعة العناصر التي تنتمي لكلا المجموعتين معا.

$$A \cap B = \{x; x \in A \wedge x \in B\}$$

$$x \in A \cap B \Leftrightarrow x \in A \wedge x \in B$$

$$x \notin A \cap B \Leftrightarrow x \notin A \vee x \notin B$$




**Example:** Let $A = [2,9], \; B = (5,14], \quad C = (8,12)$

Find $A \cap B, \; A \cap \emptyset, \; B \cap B, \; B \cap C, \; A \cap C, \; (A \cap B) \cup (B \cap C)$

**Definition:** Generalization of the intersection تعميم التقاطع Let $A_1, A_2, \ldots, A_n, A_{n+1} \ldots$ be any sets. Then:

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \ldots \cap A_n = \{x, x \in A_i \; \forall i = 1,2, \ldots, n\}$$

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In general,

$$\bigcap_{i=1}^{\infty} A_i = A_1 \cap A_2 \cap \ldots \cap A_n \cap A_{n+1} \ldots = \{x, x \in A_i \; \forall i\}$$

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**Example:** Let X = $\{A_i\ ;\ A_i = \{1,2,3,\ldots, i\ \};\ i \in$ N$\}$

Find $(\bigcup_{i=1}^{5} A_i) \cap (\bigcap_{i=1}^{5} A_i)$

**Solution:** $A_1 = \{1\}$, $A_2 = \{1,2\}$ ,… $A_5 = \{1,2,3,4,5\}$

$\bigcup_{i=1}^{5} A_i = A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 = \{1,2,3,4,5\}$

$\bigcap_{i=1}^{5} A_i = A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5 = \{1\}$

$(\bigcup_{i=1}^{5} A_i) \cap (\bigcap_{i=1}^{5} A_i) = \{1,2,3,4,5\} \cap \{1\} = \{1\}$

**Example:** (H. W.) Let $X = \{A_n\ ;\ A_n = \{n^3 + 1\};\ n \in Z\}$

Find $\bigcap_{n=-3}^{0} A_n$ , $\bigcap_{n=1}^{\infty} A_n$ , $P(\bigcup_{n=1}^{3} A_n)$

**Example:** (H. W.) Let $X = \{A_n\ ;\ A_n = \{n-2, n-1, n\};\ n \in N\}$

Find $\bigcap_{n=1}^{\infty} A_n$ , $\bigcup_{n=1}^{\infty} A_n$

**Theorem:** Let A, $B$ and $C$ any three sets. Then:

1. $A \cap \emptyset = \emptyset$ **(H.W.)** (Domination law)
2. $A \cap A = A$ **(H.W.)** (Idempotent law)
3. $A \cap U = A$ (Identity law)
4. $(A \cap B) \cap C = A \cap (B \cap C)$
5. $A \cap B = B \cap A$**(H.W.)**
6. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
7. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ **(H.W.)**
8. $A \cap B \subseteq A$ , $A \cap B \subseteq B$ **(H.W.)**
9. $A \subseteq B \iff A \cap B = A$
10. P(A) ∩ P(B) = P(A ∩ B) **(H.W.)**

**Proof 3:** T. P $A \cap U \subseteq A \wedge A \subseteq A \cap U$

Assume that $x \in A \cap U \Longrightarrow x \in A \wedge x \in U$

$\Longrightarrow x \in A$

$\therefore A \cap U \subseteq A$ …………..(1)

Let $x \in A$ T.P $x \in A \cap U$

$x \in A \Longrightarrow x \in A \wedge x \in U$ [A ⊆ U]

$\Longrightarrow x \in A \cap U$ …………..(2)

From (1) & (2), $A \cap U = A$

**Proof 4:**

Let $x \in (A \cap B) \cap C \Leftrightarrow x \in (A \cap B) \wedge x \in C$ (def. of ∩)

$\Leftrightarrow (x \in A \wedge x \in B) \wedge x \in C$ (def. of ∩)

$\Leftrightarrow x \in A \wedge (x \in B \wedge x \in C)$ (∧ is assoc.)

$\Leftrightarrow x \in A \wedge x \in (B \cap C)$ (def. of ∩)

$\Leftrightarrow x \in A \cap (B \cap C)$ (def. of ∩)

$\therefore (A \cap B) \cap C = A \cap (B \cap C)$

**Proof 6:**

Let $x \in A \cap (B \cup C) \Leftrightarrow x \in A \wedge x \in B \cup C$ (def. of ∩)

$\Leftrightarrow x \in A \wedge (x \in B \vee x \in C)$ (def. of ∪)

$\Leftrightarrow (x \in A \wedge x \in B) \vee (x \in A \wedge x \in C)$ (distribute ∧ on v)

$x \in (A \cap B) \vee x \in (A \cap C) \Leftrightarrow x \in (A \cap B) \cup (A \cap C)$ (def. of ∪)

**Proof 9:** T. P $A \subseteq B \Leftrightarrow A \cap B = A$

(⟹) Suppose $A \subseteq B$ T.P $A \cap B \subseteq A \wedge A \subseteq A \cap B$

Let $x \in A \cap B \Longrightarrow x \in A \wedge x \in B$ (def. of ∩)

$\Longrightarrow x \in B \wedge x \in B \Longrightarrow x \in B$

$\therefore A \cap B \subseteq A$ ………….(1)

Let $x \in A \Longrightarrow x \in A \wedge x \in B$ [A ⊆ B]

$\Rightarrow x \in A \cap B$

$\Rightarrow \subseteq B \cap C$ ...... (2)

From (1) & (2), A ∩ B = A

(⇐) Let $A \cap B = A$ T. P $A \subseteq B$ (H. W.)

## <u>3. The Complement</u> المتممة أو المكملة

Let $U$ be a universal set and $A$ be any subset of $U$. The complement of a set $A$, denoted by $A^C$ , is the set of elements which belong to $U$ but do not belong to $A$.

متممة المجموعة A هي مجموعة العناصر التي تنتمي للمجموعة الشاملة ولا تنتمي للمجموعة A .

$A^c = \{x, x \in U \wedge x \notin A\} = U \backslash A$

$x \in A^c \Leftrightarrow x \notin A$

$x \in A \Leftrightarrow x \notin A^c$

**<u>Example:</u>** Let U = Z, A = $\{-1,0,1\}$. Find $A^C$ .

Solution: $A^C$ = Z \A

**<u>Example:</u>** Let U = [0,8),A = {1,2}. Find $A^C$ .

$A^C$ = [0,1) ∪ (1,2) ∪ (2,8)

**<u>Example:</u>** Let $U = \{1,2, \dots ,10\}$,

A = $\{x \in N : 1 \leq x \leq 3\}$= {1,2,3}

B =$\{x \in N : 8 \leq x \leq 10\}$ = {8,9,10}

$C = \{x \in N : 1 \leq x \leq 2\} = \{1,2\}$

Find $A^C$ , $B^C$ , $C^C$ , $(A \cup B)^C$ , $(A \cap C)^C$ , $(C \cup B)^C$

..........................................................................................................................................

**Theorem:** Let $A$ and $B$ any two sets. Then:

1. $\emptyset^C = U$, $U^C = \emptyset$

2. $A^C \cap A = \emptyset$ **(H. W.)**, $A^C \cup A = U$ , $(A^C)^C = A$ **(H. W.)**

3. $(A \cup B)^C = A^C \cap B^C$

4. $(A \cap B)^C = A^C \cup B^C$ (H. W.)

5. $A \subseteq B \Leftrightarrow B^C \subseteq A^C$ (H. W.)

6. $A \subseteq B \Leftrightarrow A \cap B^C = \emptyset$

**Proof 1:** T. P $\emptyset^C = U$

Assume that $\emptyset^C \neq U$ برهان غير مباشر

$\exists x, x \in \emptyset^C \wedge x \notin U$

$x \in \emptyset^C \Longrightarrow x \in U \wedge x \notin \emptyset$(def. of $\emptyset^C$)

$\Longrightarrow x \in U$ C! تناقض مع الفرض

$\therefore \emptyset^C = U$

**Proof 1:** T. P $U^C = \emptyset$

Assume that $U^C \neq \emptyset$ برهان غير مباشر

$\exists x, x \in U^C \wedge x \notin \emptyset$

$x \in U^C \Longrightarrow x \in U \wedge x \notin U$ C! تناقض مع الفرض

$\therefore U^C = \emptyset$

**Proof 2:** T. P $A^C \cup A = U$

Assume that $A^C \cup A \neq U$ برهان غير مباشر

$\exists x, x \in A^C \cup A \wedge x \notin U$ ……(*)

$x \in A^C \cup A \Longrightarrow x \in A^C \vee x \in A$ (def. of $\cup$)

If $x \in A^C \Longrightarrow x \in U \wedge x \notin A$(contradiction with $x \notin U$ in *)

If $x \in A \Longrightarrow x \in U \wedge x \notin A^C$(contradiction with $x \notin U$ in *)

$\therefore A^C \cup A = U$

**Proof 3:** T. P $(A \cup B)^C = A^C \cap B^C$

$x \in (A \cup B)^C \Leftrightarrow x \notin (A \cup B)$ (def of complement)

$\Leftrightarrow x \notin A \wedge x \notin B)$ (def of $\cup$)

$\Leftrightarrow x \in A^C \wedge x \in B^C$ (def. of $A^C$ )

$\Leftrightarrow x \in A^C \cap B^C$ (def. of $\cap$)

**Proof 6: ($\Longrightarrow$)** Let $A \subseteq B$ T. P $A \cap B^C = \emptyset$

Assume that $A \cap B^C \neq \emptyset$ برهان غير مباشر بالتناقض

$\exists x, x \in A \cap B^C \Longrightarrow x \in A \wedge x \in B^C$

$\Longrightarrow x \in B \wedge x \in B^C$ ( by hypo. $A \subseteq B$) $\Longrightarrow$ تناقض

$\therefore A \cap B^C = \emptyset$

($\Longleftarrow$) Let $A \cap B^C = \emptyset$ T. P $A \subseteq B$

Let $x \in A \Longrightarrow x \notin B^C$ ( by hypo. $A \cap B^C = \emptyset$)

$\Longrightarrow x \in B$ (def. of $B^C$)

$\therefore A \subseteq B$

## 4. Difference or relative complement : الفضلة أو الفرق

Let $A$ and $B$ are two sets. The difference between $A$ and $B$, denoted as $A - B$ or $A\backslash B$, is the set of elements which belong to $A$ but do not belong to $B$.

يقال لمجموعة العناصر المنتمية إلى A وغير منتمية إلى B بأنها فضلة A على B.

$A - B = \{x, x \in A \wedge y \notin B\} = A\backslash B$

$x \in A - B \Leftrightarrow x \in A \wedge x \notin B$

$x \notin A - B \Leftrightarrow x \notin A \vee x \in B$

**Example:** Let $A = \{x \in Z: x \geq 1\} = N$

$B = \{x \in R : x^2 + 3 = 0\} = \emptyset$

$C = \{x \in Z: -3 < x \leq 4\} = \{-2, -1, \dots, 4\}$

$D = \{x \in O: y^2 - 9 = 0\} = \{3, -3\}$

Find A\D , B – B , C\(A ∩ D) , (B ∪ A)\C , (C ∪ A)\(B ∩ D) **(H. W.)**

**Solution**: $A\backslash D = N\backslash\{3, -3\} = N\backslash\{3\}$

**Theorem:** Let A,B and C any three sets. Then:

1. $A\backslash A = \emptyset,\ A\backslash U = \emptyset$ (H. W.)

2. $A\backslash\emptyset = A,\ ,\ \emptyset\backslash A = \emptyset$ (H. W.)

3. $\backslash B \subseteq A$ , $B\backslash A \subseteq B$ (H.W)

4. $A \subseteq B \Leftrightarrow A\backslash B = \emptyset$ (H. W.)

5. $A\backslash B \cap B = \emptyset$

6. $A \cap B = \emptyset \Leftrightarrow A\backslash B = A \wedge B\backslash A = B$

7. $A\backslash(B \cup C) = A\backslash B \cap A\backslash C$ (H. W.)

8. $A\backslash(B \cap C) = A\backslash B \cup A\backslash C$

9. $\backslash A^C = A$ , $A^C\backslash A = A^C$ (H. W.)

**Proof 5:** Assume that $A\backslash B \cap B \neq \emptyset$ برهان غير مباشر

$\exists x, x \in A\backslash B \cap B \Longrightarrow x \in A\backslash B \wedge x \in B$

$\Longrightarrow (x \in A \wedge x \notin B) \wedge x \in B$ (def. of difference)

$\Longrightarrow x \in A \wedge (x \notin B \wedge x \in B)$ ($\wedge$ is asso.)

$\Longrightarrow x \in A \wedge F \Longrightarrow$ تناقض $(P \wedge F = F)$

$\therefore A\backslash B \cap B \neq \emptyset$

**Proof 6:** ($\Longrightarrow$) Let $A \cap B = \emptyset$ T. P $A\backslash B = A \wedge B\backslash A = B$

Let x $\in A\backslash B \Leftrightarrow x \in A \wedge x \notin B$ (def. of \)

$\Leftrightarrow x \in A \wedge T$ (by hypo.)

$\Leftrightarrow x \in A$ ($P \wedge T = P$)

$\therefore A\backslash B = A$

Similarly, one can prove that $B\backslash A = B$

($\Leftarrow$) Let $A\backslash B = A \wedge B\backslash A = B$ T. P $A \cap B = \emptyset$

Assume that $A \cap B \neq \emptyset$ برهان بالتناقض

$\exists$x, x $\in A \cap B \Rightarrow$ x $\in A \wedge$ x $\in B$ (def. of $\cap$)

$\Rightarrow \exists$x, x $\in A\backslash B \wedge$ x $\in B$ ($A\backslash B = A$)

$\Rightarrow \exists x, (x \in A \wedge x \notin B) \wedge x \in B$

$\Rightarrow \exists x, x \in A \wedge (x \notin B \wedge x \in B)$ ($\wedge$ is asso.)

$\Rightarrow \exists x, x \in A \wedge F \Rightarrow P \wedge F = F$ تناقض

$\therefore A \cap B = \emptyset$

**Proof 8:** T. P $A\backslash(B \cap C) = A\backslash B \cup A\backslash C$

let $x \in A\backslash(B \cap C) \Leftrightarrow x \in A \wedge$ x $\notin (B \cap C)$ (def. of \)

$\Leftrightarrow x \in A \wedge$ ( x $\notin B \vee$ x $\notin C$) (def. of $\cap$)

$\Leftrightarrow$ ($x \in A \wedge$ x $\notin B$) $\vee$ ( $x \in A \wedge$ x $\notin C$)

$\Leftrightarrow x \in A\backslash B \vee x \in A\backslash C$ (def. of \)

$\Leftrightarrow x \in A\backslash B \cup A\backslash C$

$\therefore A\backslash(B \cap C) = A\backslash B \cup A\backslash C$

## 5. Symmetric Difference الفرق التناظري

The symmetric difference between two sets $A$ and $B$ is denoted by $A\Delta B$ and is defined as:

$$A\Delta B = (A\backslash B) \cup (B\backslash A)$$

$$= (A \cup B) \backslash (A \cap B)$$

**Example:** Let $A = \{x \in E: -8 \leq x < 9\} = \{-8, -6, \dots, 0, 2, \dots 8\}$

$B = \{1,2,4,6\}$ Find $A\Delta B$

**Solution:**
$A\Delta B = (A\backslash B) \cup (B\backslash A) = \{-8, -6, \dots 0, 6, 8\} \cup \{1\}$

**Theorem:** Let $A, B$ and $C$ any three sets. Then:

1. $A\Delta A = \emptyset$ ($\boldsymbol{H}$.W.), $A \Delta \emptyset = A$

2. $A\Delta B = B\Delta A$

3. $A\Delta B = \emptyset \iff A = B$

4. $A\Delta(B\Delta C) = (A\Delta B)\Delta C$

5. $A \cap (B\Delta C) = (A \cap B)\Delta(A \cap C)$

**Proof 1:** T. P $A \Delta \emptyset = A$

Let $x \in A \Delta \emptyset \iff x \in (A\backslash\emptyset \cup \emptyset\backslash A)$ (def. of Δ)

$\iff x \in A\backslash\emptyset \vee x \in \emptyset\backslash A$

$x \in$ A $\cup \emptyset$ $(A\backslash\emptyset = A, \emptyset\backslash A = \emptyset)$

$\iff x \in A$

$\therefore A \Delta \emptyset = A$

**Proof 2:** T. P $A\Delta B = B\Delta A$

Let x $\in A\Delta B \iff x \in (A\backslash B) \cup (B\backslash A)$ (def. of Δ)

$\iff x \in (B\backslash A) \cup (A\backslash B)$ (∪ is comm.)

$\iff x \in B\Delta A$

$\therefore A\Delta B = B\Delta A$

**Proof 3:** $A\Delta B = \emptyset \iff A = B$

($\Longrightarrow$) let $A\Delta B = \emptyset$ T. P $A = B$

Suppose $A \neq B \Longrightarrow \exists x, x \in A \wedge x \notin B$

$\Longrightarrow \exists x, x \in A \setminus B$ (def. of \)

$\Longrightarrow \exists x,\ x \in A\setminus B \ \vee\ x \in B\setminus A \quad (T \vee P = T)$

$\Longrightarrow \exists x,\ x \in (A\setminus B \cup B\setminus A)$ (def. of ∪)

$\Longrightarrow \exists x, x \in A\Delta B \neq \emptyset$ contradiction تناقض

$\therefore A = B$

( ⇐) suppose $A = B$ T. P $A\Delta B = \emptyset$

$A\Delta B = A\setminus B \cup B\setminus A$ (def. of Δ)

$= A\setminus A \cup A\setminus A \quad (A = B)$

$= \emptyset \cup \emptyset = \emptyset$