2- Gamma distribution

Def : If $\propto > 0$, we define the gamma function $\Gamma_{\propto} = \int_0^{\propto} y^{\propto-1} e^{-y} dy$

Properties of gamma function :

1- $\Gamma_{(a+1)} = \propto \Gamma_{\propto}$
2- $If\ \ \propto is\ positive\ integer\ then\ \Gamma_{(\propto+1)} = \propto !$
3- $\Gamma_{\propto} = 2\int_0^{\infty} x^{2\propto-1}\, e^{-x2} dx$
4- $\Gamma_{\frac{1}{2}} = \sqrt{\pi}$

Note :

1- If $\propto = 1 \ \Rightarrow \Gamma(\propto) = \int_0^{\propto} y_e^{\propto-1}\, dy = 1$
2- If $\ \propto > 1 \Rightarrow \pi_{(x)} = (a-r)\int_0^{\propto} y^{\propto-2} e^{-y} dy = (\propto -1)\pi^{(\propto-1)} = \ \pi(\propto) = (\propto -1)!$

Def : the continuous r.v x is said to have a gamma distribution with parameters $\propto$, $\beta > 0$ denoted as x~G ($\propto, \beta$) if the p. d. f of x is $f(x) = \begin{cases} \frac{1}{\Gamma \propto \beta^{\propto}} x^{\propto -1}\ e^{-x/\beta} & ,x>0 \\ 0 & 0.w \end{cases}$

Properties of gamma distribution: -

1- The m. g. f of the distribution is $M_x(t) = (1-\beta t^{-\propto})$, $t\ < \frac{1}{\beta}$

Proof : $M_x(t) = E(e^{tx}) = \int_0^{\infty} e^{tx}\, f(x)dx = \ \int_0^{\infty} e^{tx} \frac{1}{\Gamma_{\propto}\, \beta^{\propto}} x^{\propto -1}\, e^{\frac{-x}{\beta}} dx$

$$= \frac{1}{\Gamma_{\propto}\, \beta^{\propto}} \ \int_0^{\infty} x^{\propto-1} e^{tx-\ \frac{x}{\beta}}\, dx = \frac{1}{\Gamma_{\propto}\, \beta^{\propto}} \int_0^{\infty} x^{\propto-1}\ e^{\frac{\beta tx - x}{\beta}}\ dx$$

$$= \frac{1}{\Gamma_{\propto}\, \beta^{\propto}} \int_0^{\infty} x^{\propto-1}\ e^{-x(\frac{1-\beta t}{\beta})}\ dx$$

Patting $\quad y=x(\frac{1-\beta t}{\beta}) \ \Rightarrow x = \frac{\beta}{1-\beta t}\, y, dx = \frac{\beta}{1-\beta t} dy$

$$M_x(t) = \frac{1}{\Gamma_{\propto}\, \beta^{\propto}} \int_0^{\infty} \left[\frac{\beta y}{1-\beta t}\right]^{\propto-1} \ e^{-y}\ \frac{\beta}{1-\beta t}\ dy$$

$$= \frac{1}{\Gamma_{\propto}\, \beta^{\propto}} \quad \frac{\beta^{\propto}}{(1-\beta t)^{\propto}} \quad \int_0^{\infty} e^{-y}\ y^{\propto -1}\ dy$$

$$= \frac{1}{\Gamma_{\propto}\, \beta^{\propto}} \left(\frac{\beta}{1-\beta t}\right)^{\propto} \Gamma_{\propto} \ = \frac{1}{(1-\beta t)^{\propto}} = (1-\beta t)^{-\propto}$$

2- $M_x = \mathrm{E(x)} = \propto \beta$

3- $\delta \mathrm{x}^2 = \mathrm{ver\ (x)}. \propto \mathrm{B}^2$ prove it

4- the k^{th} $moment\ \ about\ origin\ is\ E(x^k) = \frac{\beta^k\, \Gamma_{\propto + k}}{\Gamma_\propto}\ , k = 1,2,3, \ldots$

proof : $\mathrm{E}(x^k) = \int_0^\infty x^k\, f(\propto) dx = \int_0^\infty x^k\, \frac{1}{\Gamma_\propto\, \beta^\propto}\, x^{\propto - 1}\, e^{\frac{-x}{\beta}}\, dx$

$= \frac{1}{\Gamma_\propto\, \beta^\propto} \int_0^\infty x^{k + \propto - 1}\ \ e^{-x/\beta}\, dx$

Let $\mathrm{y} = \frac{x}{\beta} \Rightarrow x = \beta y \Rightarrow dx = \beta\, d\, y$

$\mathrm{E}(x^k) = \frac{1}{\Gamma_\propto\, \beta^\propto} \int_0^\infty (\beta y)^{\propto + k - 1}\, e^{-y}\, (\beta\, d\, y)$

$= \frac{1}{\Gamma_\propto\, \beta^\propto}\, \beta^{\propto + k} \int_0^\infty y^{\propto + k - 1}\, e^{-y} dy$ $\int_0^\infty y^{\propto - 1}\, e^{-y} dy = \Gamma_{(\propto)}$

$= \frac{1}{\Gamma_\propto\, \beta^\propto}\, \beta^\propto \beta^k \Gamma_{(\propto + k)} = \frac{\beta^k \Gamma_{\propto + k}}{\Gamma_\propto}$ $, k = 1,2, \ldots.$

Ex: use the formula of (x^k) , to find μ_x , δx^2

Solution : putting $k = 1\ then$

$\mathrm{E(x)} = M_x\ \frac{\beta \Gamma_{\propto + 1}}{\Gamma_\propto} = \frac{\beta \propto \Gamma_\propto}{\Gamma_\propto} = \propto \beta$

Putting $k = 2\ , we\ get$

$\mathrm{E}(x^2) = \frac{\beta^2\ \Gamma_{(\propto + 2)}}{\Gamma_\propto} = \frac{\beta^2 (\propto + 1) \Gamma_{(\propto + 1)}}{\Gamma_\propto}$

$= \frac{\beta^2 (\propto + 1) \propto \Gamma_\propto}{\Gamma_\propto} = \beta^2 \propto (\propto + 1)$

$\delta x^2 = var\ (x) = E(x^2) - [E(x)]^2$

$= \beta^2\ (\propto + 1) \propto\ - \beta^2\ \ \propto^2$

$= \beta^2\ \propto^2 + \beta^2 \propto - \beta^2\ \propto^2 = \propto \beta^2$

3- the chi square distribution

The chi square distribute on is as pecial case of gamma distribution which $\propto = \frac{r}{2}$ and B=2 where ris positive integer . Hence the p.d.f of the r. v. x is

$$f(x) = \left[\begin{array}{ll} \frac{1}{\Gamma_{\frac{r}{2}} 2^{r/2}} \; x^{\frac{r}{2}-1} \; e^{\frac{-x}{2}}, & x > 0 \\ 0 & 0.w \end{array}\right.$$

and we write x∼ $x^2\ (r)$ where r is the number of degrees freedom representing the parameter of the distribution .

properties :-

the properties of chi square dist are the same of properties of gamma dist when $\propto = \frac{r}{2}, \beta = 2$ that is :-

1) $M_x(t) = (1-2t)^{-r/2}$
2) $M_x = E(x) = \frac{r}{2} \cdot 2 = r$
3) Var(x) = $\delta x^2 = \left(\frac{r}{2}\right) 2^2 = 2r$
4) E(x^k) $= \frac{2^k\, \Gamma_{(\frac{r}{2}+k)}}{\Gamma_{\frac{r}{2}}}, k = 1,2,\ldots.$

4-Beta distribution

Def :- If $\propto > 0, \beta > 0$, we define the Beta function as

$$\beta\,(\propto, \beta) = \int_0^1 x^{\propto-1}(1-x)^{\beta-1}\,dx$$

It can be show that $\beta\,(\propto, \beta) = \frac{\Gamma_\propto \Gamma_B}{\Gamma_{\propto+rs}}, \propto, \beta > 0$

Def :- the continuous r. v. x is said to have a Beta distribution denoted as x ∼ β (∝ , β) If the p. d. f of x is $f(x) = \begin{cases} \frac{\Gamma_{\propto+\beta}}{\Gamma_\propto \Gamma_\beta} x^{\propto-1} (1-x)^{\beta-1}, & 0 < x < 1 \\ 0 & 0.w \end{cases}$

Properties :-

1) The k^{th} moment a bout origin is $E\,(x^k) = \frac{\Gamma(_{k+\propto)}\Gamma_{(\propto+B)}}{\Gamma_\propto \Gamma_{(k+\propto+B)}}, |k = 1, 2, \ldots$

Proof: $E\,(x^k) = \int_0^1 x^k\, f(x)dx = \frac{\Gamma_{\propto+B}}{\Gamma_\propto \Gamma_B}\ \int_0^1 x^{\propto 1-\,k-1}\,(1-x)^{\beta-1}dx$

$= \frac{\Gamma_{\propto+\beta}}{\Gamma_\propto \Gamma_\beta}\ \frac{\Gamma_{\propto+k}\Gamma_\beta}{\Gamma_{\propto+k+\beta}}\quad \left[\beta(\propto,\beta) = \frac{\Gamma_\propto \Gamma_\beta}{\Gamma_{\propto+\beta}}\right]$

$= \frac{\Gamma_{\propto+B}\,\Gamma_{\propto+k}}{\Gamma_\propto\,\Gamma_{k+\propto+\beta}}$

2) From the above formula , the mean and variance of the distribution can be derive as follows :- putting k=1 we obtain $E(x) = M_x = \frac{\Gamma_{\propto+1}\Gamma_{\propto+\beta}}{\Gamma_\propto \Gamma_\propto+\beta+1} = \frac{\propto\Gamma_\propto \Gamma_{\propto+\beta}}{\Gamma_\propto(\propto+\beta)\Gamma_{\propto+\beta}}\ [since\ \Gamma_{\propto+1} = \propto \Gamma_\propto]$

$M_x = \frac{\propto}{\propto+\beta}$

Putting k=2 we get $E(x^2) = \frac{\Gamma_{\propto+2}\Gamma_{\propto+\beta}}{\Gamma_\propto \Gamma_\propto+\beta+2} = \frac{(\propto+1)\propto\Gamma_\propto \Gamma_{\propto+}\beta}{\Gamma_\propto(\propto+\beta+1)(\propto+\beta)\Gamma_{\propto+\beta}}$

$= \frac{\propto(\propto+1)}{(\propto+\beta+1)(\propto+\beta)}$

Var (x) = $\int_x^2 = E(x^2) - [E(x)]^2$

$= \frac{\propto(\propto+1)}{(\propto+\beta+1)(\propto+\beta)} - \frac{\propto^2}{(\propto+\beta)^2} = \frac{\propto(\propto+1)(\propto+\beta)-\propto^2(\propto+\beta+1)}{(\propto+\beta)^2(\propto+\beta+1)}$

$= \frac{\propto^3+\propto^2\beta+\propto^2+\propto\beta-\propto^3-\propto^2\beta-\propto^2}{(\propto+\beta)^2(\propto+\beta+1)} = \frac{\propto\beta}{(\propto+\beta)^2(\propto+\beta+1)}$

$\therefore \int_x^2 = \frac{\propto\beta}{(\propto+\beta)^2(\propto+\beta+1)}$

5- The normal distribution

A continuous r.v x is said to have normal distribution with parameters M , δ^2 denote , as x~N(M, δ^2) if the p. d. f of x is $f(x) = \frac{1}{\sqrt{2\pi\,\delta^2}}\ e^{\frac{-1}{2}(\frac{x-M}{\delta})^2}$ -∞ $< x < \infty$

<u>Proportion :-</u>

1) The M. g. f of normal dist. Is $M_x(t) = e^{Mt\,\frac{\delta^2 t2}{2}}$

Proof :- $M_x(t) = E(e^{tx}) = \frac{1}{\sqrt{2\pi\delta^2}} \int_{-\infty}^{\infty} e^{+x}\, e^{\frac{-1}{2}\ (\frac{x-M}{\delta})^2}\, dx$