Theorem(3) :- If xi , i= 1,2,…,n distributed as $\sim N$ (0,1) then $\sum_{i=1}^{n} x_i^2 \sim x^2(n)$

Proof :- let y= $\sum_{i=1}^{n} x_i^2$, then $\mu_y(t) = E(e^{ty})$
$M_y(t) = E\left(e^{+(x_1^2+x_2^2+\cdots+x_n^2}\right)$

= E$(e^{tx_1^2})$ $E\left(e^{tx_2^2}\right) \ldots E(e^{tx_n^2})$

Since each of x_1, x_2,…,xn ~N(0,1) then each of $x_1^2, x_2^2, \ldots, x_n^2 \sim x^2(1)$ (theories 2)
$M_y(t) = (1-2t)^{-1/2} (1-2t)^{-1/2} \ldots (1-2t)^{-1/2}$ n-terms
$M_y(t) = \left[((1-2t)^{-1/2}\right]^n = (1-2t)^{-n/2}$
y= $\sum_{i=1}^{n} xi^2 \sim x^2(n)$

(6) the students t distribution

Let the r.v W~N(0,1) and the r.v. V$\sim x^2(r)$, where W and V are stochastically independent . then T=$\frac{w}{\sqrt{\frac{v}{r}}}$ has students t distribution with p. d. f given by

$$g(t) = \frac{\Gamma_{(r+1)/2}}{\sqrt{\pi r}\,\Gamma_{\frac{r}{2}}} \quad \frac{1}{(1+\frac{t^2}{r})^{2\frac{(r+1)}{2}}} \quad , -\infty < t < \infty$$

Proof :- the joint p. d. f. of W and V is $\phi(w,v) = \frac{1}{\sqrt{2\pi}} e^{-w^2/2} \frac{1}{\Gamma_{\frac{r}{2}} 2^{r/2}} (v)^{\frac{r}{2}-1} e^{\frac{-v}{2}}$

$-\infty < w < \infty , 0 < v < \infty$

Let : $t = \frac{w}{\sqrt{\frac{v}{r}}}$ and u=v define $a(1-1)$ transformation mapping the space $\{(w,v)_i - \infty < w < \infty , 0 < v < \infty\}$ onto the space $\{(t,u)_i - \infty < t < \infty , 0 < u < \infty\}$

$$w = t\frac{\sqrt{u}}{\sqrt{r}} , v = u , J = \begin{vmatrix} \frac{dw}{dt} & \frac{dw}{du} \\ \frac{dv}{dt} & \frac{dv}{du} \end{vmatrix} = \begin{vmatrix} \frac{\sqrt{u}}{\sqrt{r}} & \frac{t}{2\sqrt{ur}} \\ 0 & 1 \end{vmatrix}$$

$= \frac{\sqrt{u}}{\sqrt{r}}$

$$g(t,u) = \emptyset \left[\frac{t\sqrt{u}}{\sqrt{r}} , u\right] \ . \ |J|$$

$$= \frac{1}{\sqrt{2\pi}\,\Gamma_{\frac{r}{2}}\,2^{r/2}}\;u^{\frac{r}{2}-1}\,e^{\frac{-u}{2}}\,e^{\frac{-t^2u}{2r}}\,\frac{\sqrt{u}}{\sqrt{r}}$$

$$= \frac{1}{\sqrt{2\pi}\,\Gamma_{\frac{r}{2}}\,2^{r/2}}\;u^{\frac{r}{2}-1}\,e^{\frac{-u}{2}\left[1+\frac{t^2}{r}\right]}\frac{\sqrt{u}}{\sqrt{r}}$$

$-\infty < t < \infty\,, 0 < u < \infty$ Now $g(t) = \int_0^\infty g(t,u)du$

$$=\int_0^\infty \frac{1}{\sqrt{2\pi}\,\Gamma_{\frac{r}{2}}\,2^{r/2}}\;u^{\frac{r}{2}-1}\,e^{\frac{-u}{2}\left[1+\frac{t^2}{r}\right]}\;\frac{\sqrt{u}}{\sqrt{r}}\,du$$

Putting $Z=\frac{u}{2}\left[1+\frac{t^2}{r}\right] \Rightarrow u\left[1+\frac{t^2}{r}\right] = 2Z$

$$u = \frac{2Z}{1+\frac{t^2}{r}} \Rightarrow du = \frac{2}{1+\frac{t^2}{r}}\,dZ$$

$$= \int_0^\infty \frac{1}{\sqrt{2\pi}\,\Gamma_{\frac{r}{2}}\,2^{r/2}\,r^{1/2}}\;\left(\frac{2Z}{1+\frac{t^2}{r}}\right)^{\frac{r}{2}-1} e^{-Z}\;\frac{\sqrt{2Z}}{\sqrt{1+\frac{t^2}{r}}}\;\frac{2}{1+\frac{t^2}{r}}\,dZ$$

$$= \frac{1}{\sqrt{2\pi r}\,\Gamma_{\frac{r}{2}}\,2^{r/2}}\;\frac{2^{\frac{r}{2}-1}}{(1+\frac{t^2}{r})^{\frac{r}{2}-1}}\;\frac{\sqrt{2}}{(1+\frac{t^2}{r})^{1/2}}\;\frac{2}{(1+\frac{t^2}{r})}\;\int_0^\infty Z^{\frac{r}{2}-1}Z^{1/2}e^{-Z}dZ$$

$$= \frac{1}{\sqrt{2\pi r}\,\Gamma_{\frac{r}{2}}}\;\frac{\sqrt{2}}{(1+\frac{t^2}{r})^{\frac{r+1}{2}}}\;\int_0^\infty Z^{\frac{r}{2}+\frac{1}{2}-1}e^{-Z}dZ$$

$$g(t) = \frac{\Gamma_{\frac{r+1}{2}}}{\sqrt{\pi r}\quad \Gamma_{\frac{r}{2}}}\;\frac{1}{(1+\frac{t^2}{r})^{\frac{r+1}{2}}}\quad , -\infty < t < \infty$$

The mean and variance of distribution

The mean of students t , distribution is :- $E(t) = E\left(\frac{w}{\sqrt{\frac{r}{r}}}\right) = E(W)E\left(\frac{1}{\sqrt{\frac{r}{r}}}\right)$

[since ,w. v are indep.] but W~N(0,1) , E(W)=0 hence E(t)=0 , E $\left(\frac{1}{\sqrt{\frac{r}{r}}}\right)$ = 0

the variance of distribution is derived as follows var (t) = E(t^2) $-[E(t)]^2 = E(t^2)$

var (t) = E$\left[\frac{w}{\sqrt{\frac{r}{r}}}\right]^2 = E\left[\frac{w^2}{\frac{v}{r}}\right] = E(w^2)E\left(\frac{1}{\frac{v}{r}}\right)$

since W~N(0,1) then E(w^2) $= Var(w) + [E(w)]^2$ = 1

var(t)= 1.E$\left(\frac{1}{\frac{v}{r}}\right) = r\ E\ (\frac{1}{V})$

but V~$x^2(r)$ then E$\left(\frac{1}{V}\right) = \int_0^\infty \frac{1}{V}\ \frac{1}{\Gamma_{\frac{r}{2}} 2^{r/2}}\ v^{\frac{r}{2}-1} e^{\frac{-v}{2}} \delta v$

$= \frac{1}{\Gamma_{\frac{r}{2}} 2^{r/2}} \int_0^\infty v^{-1}\ v^{\frac{r}{2}-1} e^{\frac{-v}{2}}\ \delta v = \frac{1}{\Gamma_{\frac{r}{2}} 2^{r/2}} \int_0^\infty v^{\frac{r}{2}-2} e^{\frac{-v}{2}}\ \delta v$

Putting Z=$\frac{v}{2}$ $\Rightarrow v = 2Z \Rightarrow \delta v = 2dZ$ $E\left(\frac{1}{V}\right) =$
$\frac{1}{\Gamma_{\frac{r}{2}} 2^{r/2}} \int_0^\infty (2Z)^{\frac{r}{2}-2}\ e^{-Z}\ 2dZ$

$= \frac{2^{\frac{r}{2}-2}\quad 2}{\Gamma_{\frac{r}{2}} 2^{r/2}} \qquad \int_0^\infty Z^{\frac{r}{2}-1-1} \quad e^{-Z}\ dZ$

$= \frac{1}{2\Gamma_{\frac{r}{2}}} \quad \Gamma_{\frac{r}{2}-1} = \frac{1}{2(\frac{r}{2}-1)\Gamma_{\frac{r}{2}-1}} \quad \Gamma_{\frac{r}{2}-1}$

$= \frac{1}{2(\frac{r}{2}-1)} = \frac{1}{r-2}$

Var(t)= r E($\frac{1}{v}$) = $\frac{r}{r-2}$

(7) The F distribution

Let $x_1 \sim x^2(n_1)$ is independent from $x_2 \sim x^2(n_2)$ then the ration f= $\frac{x_1/n_1}{x_2/n_2}$ is said to have an F distribution with n_1, n_2 degrees of freedom the p. d. f of the