[since ,w. v are indep.] but W~N(0,1) , E(W)=0 hence E(t)=0 , E $\left(\frac{1}{\sqrt{\frac{r}{r}}}\right) = 0$

the variance of distribution is derived as follows var (t) = E$(t^2) - [E(t)]^2 = E(t^2)$

var (t) = E$\left[\frac{w}{\sqrt{\frac{r}{r}}}\right]^2 = E\left[\frac{w^2}{\frac{v}{r}}\right] = E(w^2)E\left(\frac{1}{\frac{v}{r}}\right)$

since W~N(0,1) then E$(w^2) = Var(w) + [E(w)]^2$ = 1

var(t)= 1.E$\left(\frac{1}{\frac{v}{r}}\right) = r\ E\ (\frac{1}{V})$

but V$\sim x^2(r)$ then E$\left(\frac{1}{V}\right) = \int_0^\infty \frac{1}{V}\ \frac{1}{\Gamma_{\frac{r}{2}} 2^{r/2}}\ v^{\frac{r}{2}-1} e^{\frac{-v}{2}} \delta v$

$= \frac{1}{\Gamma_{\frac{r}{2}} 2^{r/2}}\ \int_0^\infty v^{-1}\ v^{\frac{r}{2}-1} e^{\frac{-v}{2}}\ \delta v = \frac{1}{\Gamma_{\frac{r}{2}} 2^{r/2}}\ \int_0^\infty v^{\frac{r}{2}-2} e^{\frac{-v}{2}}\ \delta v$

Putting Z=$\frac{v}{2}$ $\Rightarrow v = 2\text{Z} \Rightarrow \delta v = 2d\text{Z}$ $E\left(\frac{1}{V}\right) =$
$\frac{1}{\Gamma_{\frac{r}{2}} 2^{r/2}}\ \int_0^\infty (2\text{Z}\,)^{\frac{r}{2}-2}\ e^{-\text{Z}}\ 2d\text{Z}$

$= \frac{2^{\frac{r}{2}-2}\quad 2}{\Gamma_{\frac{r}{2}} 2^{r/2}} \qquad \int_0^\infty \text{Z}^{\frac{r}{2}-1-1} \quad e^{-\text{Z}}\ d\text{Z}$

$= \frac{1}{2\Gamma_{\frac{r}{2}}} \quad \Gamma_{\frac{r}{2}-1}$ = $\frac{1}{2(\frac{r}{2}-1)\Gamma_{\frac{r}{2}-1}} \quad \Gamma_{\frac{r}{2}-1}$

$= \frac{1}{2(\frac{r}{2}-1)} = \frac{1}{r-2}$

Var(t)= r E$(\frac{1}{v})$ = $\frac{r}{r-2}$

(7) The F distribution

Let $x_1 \sim x^2(n_1)$ is independent from $x_2 \sim x^2(n_2)$ then the ration f= $\frac{x_1/n_1}{x_2/n_2}$ is said to have an F distribution with $n_1\,, n_2$ degrees of freedom the p. d. f of the

distribution is :- $g(f) = \frac{\sqrt{\frac{n_1+n_2}{2}}}{\Gamma\frac{n_1}{2}\Gamma\frac{n_2}{2}} \ (\frac{n_1}{n_2})^{\frac{n_1}{2}} \ \frac{f^{\frac{n_1}{2}-1}}{\left[1+\frac{n_1}{n_2}f\right]^{\frac{n_1+n_2}{2}}}$ $, f > 0$ and we say that f~F(n_1, n_2) the mean of the distribution is given by $\mu_f = E(f) = \frac{n_2}{n_2-2}$ $, n_2 > 2$

Proof :- $E(f) = E\left(\frac{x_1/n_1}{x_2/n_2}\right) = \frac{\frac{1}{n_1}}{\frac{1}{n_2}} \ E\ \left(\frac{x_1}{x_2}\right)$

$= \frac{n_2}{n_1} \ E(x_1)\ E(\frac{1}{x_2})$

Since $x_1 \sim x_2(n_1), x_2 \sim x^2(n_2)$, then E$(x_1) = n_1$

$E(f) = \frac{n_2}{n_1}\ n_1\ E\left(\frac{1}{x_2}\right) = n_2 E(\frac{1}{x_2})$

$E\left(\frac{1}{x_2}\right) = \int_0^\infty \frac{1}{\Gamma\frac{n_2}{2}\ 2^{n_2/2}} \ \ x.\frac{1}{x_2} \ \ x_2^{\frac{n_2}{2}-1}\ e^{\frac{-x_2}{2}}\ dx_2$

$= \frac{1}{\Gamma\frac{n_2}{2}\ 2^{n_2/2}} \ \int_0^\infty x_2^{\frac{n_2}{2}-1-1} e^{\frac{-x_2}{2}}\ dx_2$

$\mathrm{Z} = \frac{x_2}{2} \Rightarrow 2\mathrm{Z} = x_2 \Rightarrow 2d\mathrm{Z} = dx_2$

$= \frac{1}{\Gamma\frac{n_2}{2}\ 2^{n_2/2}} \ \int_0^\infty (2\mathrm{Z})^{\frac{n_2}{2}-2}\ e^{-\mathrm{Z}} 2d\mathrm{Z}$

$= \frac{1}{\Gamma\frac{n_2}{2}\ 2^{n_2/2}}\ 2^{\frac{n_2}{2}-2} \quad .2\ \int_0^\infty \mathrm{Z}^{\frac{n_2}{2}-2}\ e^{-\mathrm{Z}} d\mathrm{Z}$

$= \frac{1}{2\Gamma\frac{n_2}{2}} \quad \Gamma\frac{n_2}{2}-1 \quad \quad \Gamma\frac{n_2}{2}-1$

$= \quad \frac{1}{2(\frac{n_2}{2}-1)} \quad \quad \Gamma\frac{n_2}{2}-1 \quad \quad \Gamma\frac{n_2}{2}-1 \ = \ \frac{1}{n_2-2}$

$E(f) = n_2 \ \ E\left(\frac{1}{x_2}\right) = \frac{n_2}{n_2-2}\ , n_2 > 2$

The variance of the distribution is var(f)=2 $M_f^2 \frac{n_1+n_2-2}{n_1(n_2-4)}$ $, n_2 > 4$

Problems :-

(1) Let t be ar.v with p. d. f $g(t) = c\left[1 + \frac{1}{5}t^2\right]^{-3}$ find the value of c such that the r. v. follows t distribution

(2) Let the r. v. t$\sim t(1)$show that the C.D.F of t is F(t)=0.5 + $\frac{1}{\pi} = \tan^{-1} t$

(3) Let t=$\frac{w}{\sqrt{v/r}}$, where w$\sim N(0,1)$ and v$\sim x^2(r)$ show that T^2 has an F distribution with parameters $r_1 = 1, r_2 = r$

(4) Let f has F distribution with parameters r_1, r_2 prove that $\frac{1}{f}$ has an F distribution with parameters r_2 and r_1

3)Distribution of sample mean and sample variance

(1) the distribution of$\bar{x}$:- let $x_i, i = 1,2, \dots, n$ be a r.vs from N (M,δ^2) the distribution $\bar{x}$ is derived by using the m.g.f as follows :

$M_{\bar{x}}(t) = E(e^{t\bar{x}}) = E\left(e^{t\frac{\sum x_i}{n}}\right) = E(e^{\frac{t}{n}(x_1+x_2+\dots+x_n)})$

$=$ E$(e^{\frac{t}{n}x_1})\,E\left(e^{\frac{t}{n}x_2}\right) \dots E(e^{\frac{t}{n}x_n})$

$= M_{x_1}\left(\frac{t}{n}\right)\mu_{x_2}\left(\frac{t}{n}\right) \dots M_{x_n}\left(\frac{t}{n}\right)$

Since each of $x_1, x_2, \dots, x_n$ distributed N (M,δ^2) then $M_x{}^{(t)} = e^{M\,t+\frac{\delta^2 t^2}{2}}$

$M_{\bar{x}}(t) = \left[M_x(\frac{t}{n})\right]^n = \left[e^{M\frac{t}{n}+\frac{\delta^2(\frac{t}{n})^2}{2}}\right]^n = e^{Mt+\frac{\frac{\delta^2}{n}t^2}{2}}$ which is similar to the m. g. f of normal dist. With mean M and variana $\frac{\delta^2}{n}$ $\bar{x} \sim N(M, \frac{\delta^2}{n})$ $f(\bar{x}) = \frac{1}{\sqrt{2\pi\frac{\delta^2}{n}}}\,e^{-1/2\,\frac{(\bar{x}-M)^2}{\delta^2/n}}, -\infty < \bar{x} < \infty$

for example if $x_1, x_2, \dots, x_n$ be a r.s from N (1,2) then $\bar{x} \sim N(1, \frac{2}{n})$ and

f$(\bar{x}) = \frac{1}{\sqrt{2\pi\frac{2}{n}}}\;e^{\frac{-(\bar{x}-1)^2}{2\frac{2}{n}}}, -\infty < \bar{x} < \infty$

properties :- According to theorems (1) , (2) , (3) if $\bar{x} \sim N(M, \frac{\delta^2}{n})$, then :-

1) $\frac{\bar{x}-M}{\delta/\sqrt{n}} \sim N(0,1)$
2) $(\frac{\bar{x}-M}{\delta/\sqrt{n}})^2 \sim x^2(1)$
3) $\sum_{i=1}^{k}(\frac{\bar{x}-M}{\delta/\sqrt{n}})^2 \sim x^2\,(k)$

2- Distribution of sample variance S^2

the sample variance S^2 is defined as : $S^2 = \frac{1}{n}\sum(x_i - \bar{x})^2$ consider the sum of squares $\sum(x_i - M)^2 = \sum[(x_i - \bar{x}) + (\bar{x} - M)]^2$

$=\sum(x_i - \bar{x})^2 + 2\,(\bar{x} - M)\sum(x_i - \bar{x}) + n(\bar{x} - M)^2$
$\sum(x_i - M)^2 = \sum(x_i - \bar{x})^2 + n\,(x - M)^2$

dividing both sides by δ^2 we get , $\sum(\frac{x_i-M}{\delta})^2 = \frac{n\,s^2}{\delta^2} + \left(\frac{\bar{x}-M}{\delta/\sqrt{n}}\right)^2$

since $\sum(\frac{x_1-M}{\delta})^2 \sim x^2(n)$ [theorem (3)]

Also $(\frac{\bar{x}-M}{d/\sqrt{n}})^2 \sim x^2(1)$ [property (2)]

it follows that $\frac{ns^2}{\delta^2} \sim x^2(n-1)$ (additive property of x^2 dist.)

Hence $E(\frac{ns^2}{\delta^2}) = n-1 \Rightarrow \frac{n}{\delta^2}\,E(s^2) = n-1$

$E(s^2) = \delta^2\,\frac{n-1}{n}$

$var(\frac{ns^2}{\delta^2}) = 2(n-1) \Rightarrow \frac{n^2}{\delta^4} var\,(s^2) = 2(n-1)$

$var(s^2) = \frac{2(n-1)}{n^2} \quad \delta^4$

Ex:- If $\bar{x}$ is the mean of r. v of size n from N(M,100) . find n such that $p_r(M-5 < \bar{x} < M+5) = 0.9\,s4$

Solution :- let $x_1 = M-5\,, x_2 = M+s$
$p_r[M-s < \bar{x} < M+5) = p_r[\,x_1 < \bar{x} < x_2]$

$=p_r[\frac{x_1-M}{\delta/\sqrt{n}} < \frac{\bar{x}-M}{\delta/\sqrt{n}} < \frac{x_2-M}{\delta/\sqrt{n}}]$

$= p_r\left[\frac{-5}{10/\sqrt{n}} < Z < \frac{5}{10/\sqrt{n}}\right] = p_r\left[\frac{-1}{2}\sqrt{n} < Z < \frac{1}{2}\sqrt{n}\right]$

$= p_r\left[Z < \frac{1}{2}\sqrt{n}\right) - [1 - p_r\left(Z < \frac{1}{2}\sqrt{n}\right)]$

$= \quad 2p_r\left[Z < \frac{1}{2}\sqrt{n}\right] - 1 = 0.9\ s4$

$p_r\left(Z < \frac{1}{2}\sqrt{n}\right) = \frac{1.9\ 54}{2} = 0.9\ 77$

N$(\frac{1}{2}\sqrt{n}) = 0.9\ 77$

$\frac{1}{2}\sqrt{n} = 2$ (approximately) from tables $\sqrt{n} = 4 \Rightarrow n = 16$

Ex:- let S^2 be the variance of ar.s of size six from N(μ,12) find $p_r(2.30 < S^2 < 22.2)$

Solution :- since $\frac{ns^2}{\delta^2} \sim x^2(n-1)$, then $\frac{6\ s^2}{12} = \frac{s^2}{2} \sim x^2\ (s)$

$p_r(2.30 < s^2 < 22.2) = p_r(1.1s < \frac{s^2}{2} < 11.1)$

$= p_r\left(\frac{s^2}{2} < 11.1\right) - p_r(\frac{s^2}{2} < 1.1\ s)$

= 0.9 50-0.050=0.90 (from x^2 tables)

Ex:- let $x_1, x_2, \dots, x_n$ be ar. S. from N(M,δ^2) let $\bar{x}$ and s^2 denote the mean and variance of the sample where $\bar{x}$ has s^2 are in dependent and E(s^2) $= \delta^2$, show that

$t = \frac{\sqrt{n}\ (\bar{x}-M)}{s} \sim t(n-1)$ $\quad$ $n \to \infty$ تتوزع بشكل تقربي عندما

Proof :- since x$\sim N(M, \delta^2) \Rightarrow \bar{x} \sim N(M, \frac{\delta^2}{n})$

let $Z_1 = \frac{\bar{x}-M}{\delta/\sqrt{n}} \Rightarrow Z_1 \sim N(0,1)$

let $\quad Z_2 = \frac{(n-1)s^2}{\delta^2} \Rightarrow Z_2 \sim x^2(n-1)$

$\frac{Z_1}{\sqrt{\frac{Z_2}{n-1}}} \sim t\ (n-1)$

but

$$\frac{Z_1}{\sqrt{\frac{Z_2}{n-1}}} = \frac{\frac{\bar{x}-M}{\delta/\sqrt{n}}}{\sqrt{\frac{(n-1)s^2}{(n-1)\delta^2}}} = \frac{\frac{\bar{x}-M}{\delta/\sqrt{n}}}{\frac{\sqrt{s^2}}{\sqrt{\delta^2}}} = \frac{\sqrt{n}\,(\bar{x}-M)}{\delta}\ \frac{\delta}{s}$$

t=$\frac{\sqrt{n}\,(\bar{x}-M)}{s} \sim t\ (n-1)$

تمارين التوزيعات

Problems :-

1) Given that $x \sim$Ber $(1, \frac{1}{3})$ find
   (i) the p. m. f of x, (ii)$M_x, {\delta_x}^2\ and\ \mu_x(t)$
2) The m. g. f of ar.v x is $M_x(t) = (\frac{2}{s} + \frac{1}{3}e^t)^9$
   (i) find the p.m.f of x (ii) find the mean M_x and variance ${\delta_x}^2$
   (iii)show that $p_r(M_x - 2\delta_x < x < M_x + 2\delta_x) = \sum_{x=1}^{5} x^9 (\frac{1}{3})^x\, (\frac{2}{3})^{9-x}$
3) Let $x \sim b(2,p)$ and y$\sim b(4,p)$ if $p_r(x^{\sim} \geq 1) = \frac{5}{9}$ find $p_r(y \geq 1)$
4) Let x_1, x_2 be independent r. v. s such that $x_1 \sim p(4)$ and $x_2 \sim p(6)$ let $y = x_1 + x_2$ (i) find the p.m.f of y , (ii) find $M, \delta^2 y$ (iii) find $p_r(y \leq 1)$

5) Hiven that $x \sim p(t)$ find the value of λ it is known that $f(x) = \frac{4}{x} f(x-1)$, x=1,2,….
6) Let $x_i \sim Nb(ri, p), i = 1,2, \dots, n$ show that $\sum_{i=1}^{n} x_i \sim Nb(\sum_{i=1}^{n} ri , p)$
7) Let $x \sim Nb(4,0.3)$ (i) find the p.m.f of x (ii) find $M_x, \delta x^2, M_x(t)$ (iii)let y=4+sx find $M_y, {\delta_y}^2$

تمارين التوزيعات

Problems:-

1) Let $x \sim u(0,1)$, use the transformation method to find the distribution of $y = -2Ln$ x then find $M_y, \delta y^2$
2) Given that $x_1, x_2, \dots, x_n$ are independent random variables where $x_i \sim G(\propto i, \beta), i = 1, \dots, n$ show that $y = \sum_{i=1}^{n} x_i \sim G(\sum_{i=1}^{n} \propto i , \beta)$
3) If $x_1 \sim G(2,3)$ and $x_2 \sim G(1,3)$ are two independent r.v.s (i) find the distribution of $y = x_1 + x_2$ (ii) find the mean and variance of y
4) Let $x \sim \beta(\propto, \beta)$, let $y = Ln \frac{x}{1-x}$ find $M_y(t)$
5) In each case the r. v. x follow Beta distribution , find the value of the constant c. (i) $f(x) = cx^2(1-x)^5$ (ii) $f(x) = c\,(x - x^2)^{0.5}$
6) If $\sim \beta\,(\propto, \beta)$. show that $y = (1-x) \sim \beta(\beta, \propto)$
7) If $x \sim N(0,1)$ find $E(x^{2k}), k \in \mathrm{I}^+$, then find $E(x^2)$ and $E(x^4)$
8) If $x_1, x_2, \dots, x_n$ are independent r.v.s where $x_i \sim N(M_i , \delta^2), i = 1, 2, \dots, n$ show that $y = \sum_{i=1}^{n} x_i \sim N(\sum_{i=1}^{n} m_i, \sum_{i=1}^{n} {\delta_1}^2)$
9) If $x_1 \sim N\left(M_1, {\delta_1}^2\right), x_2 \sim N(M_2, {\delta_2}^2)$, where x_1, x_2 are independent r.vs show that $y = x_1 - x_2 \sim N(M_1 - M_2, {\delta_1}^2 + {\delta_2}^2)$
10) If $x \sim x^2(n)$ and (x+y) $\sim x^2(n+m)$ where x,y are inderendent r.vs . use the m.g.f to find the distribution of y