Chapter Four:

Mappings التطبيقات

4.1Mapping (Function) التطبيق او الدالة

Let A and B be two nonempty sets. A relation f from A to B ($f \subseteq A \times B$) is called a **mapping** or **function** if each element in A is related to a unique element in B. This relation is denoted by $f: A \to B$.

الدالة او التطبيق fهي علاقة خاصة من A الى B تربط كل عنصر في المجموعة A بعنصر وحيد في المجموعة B ويرمز لهذه العلاقة بالرمز $f: A \to B$

Mathematically,

$f: A \to B$ is a mapping $\Leftrightarrow \forall x \in A\ \exists!\ y \in B\ s.t. f(x) = y$

4.2 Remark : A mapping is (generally) denoted by f, F, g, G, h, H, ….

4.3 Example : Let $A = \{1,2,3,4\}$ and $B = Z$. Which of the following relations is mapping?

$R_1 = \{(x, y) \in A \times B: y = 2x\}$ is mapping.

$R_2 = \{(1,1), (1,2), (2,0), (3, -1), (4,1)\}$ is not mapping.

$R_3 = \{(1,1), (2,0), (3,3)\}$ is not mapping.

4.4 Remark : Every function is a relation but not every relation is a function.

Mapping can be defined in another way :

4.5 Definition: Let A and B be two nonempty sets. A relation f from $A\ to\ B$ ($f \subseteq A \times B$) is called a mapping or function if it satisfies two conditions:

1.closure الانغلاق: $\forall x \in A \Rightarrow f(x) \in B$ كل عنصر في المجال صورته تنتمي للمجال المقابل

2.well-defined التعريف الجيد :

If $x_1 = x_2\ then\ f(x_1) = f(x_2)\ \forall x_1, x_2 \in A$

كل عنصر في المجال له صورة وحيدة في المجال المقابل

4.6 Example: Determine whether $f: Z \rightarrow R$ is a function or not :

a) $f(x) = \sqrt{x^2 + 1}$

1. closure: Let $\forall x, x \in Z \Rightarrow f(x) \in R$?

$\forall x, x \in Z \Longrightarrow x^2 + 1 \in Z \Longrightarrow \sqrt{x^2 + 1} \in R$

∴ closure is held

2. well defined:

$\forall x_1, x_2 \in Z, x_1 = x_2 \Longrightarrow {x_1}^2 = {x_2}^2 \Longrightarrow {x_1}^2 + 1 = {x_2}^2 + 1$

$\Longrightarrow \sqrt{{x_1}^2 + 1} = \sqrt{{x_2}^2 + 1} \Longrightarrow f(x_1) = f(x_2)$

∴ f is mapping.

b) $f(x) = \begin{cases} -x , & x \leq 1 \\ x , & x \geq 1 \end{cases}$

f is not well defined because $x = 1 \in Z$

But $f(1) = 1$ *and* $f(1) = -1$ يوجد عنصر له صورتان

3) $f(x) = \frac{1}{x}$

Closure condition is not held شرط الانغلاق غير متحقق

Let $x = 0 \in Z$ but $f(0) = \frac{1}{0} \notin R$ ∴ f is not a mapping

4.7 Example: (H.W.). Is f mapping?

1. Let $f: N \rightarrow N$ s.t. $f(x) = x/(|x| - 5)$

2. Let $f: R \rightarrow R$ s.t. $f(x) = \frac{\sqrt[3]{x}}{x-1}$

4.8 Graph of Mapping: رسم الدالة

Let A and B be two non-empty sets and $f: A \rightarrow B$. The graph of f is denoted by $Graph\ f$ and is defined as

$$Graph\ f = \{(x, y): x \in A \text{ and } y = f(x)\}$$

4.9 Definition: Let $f: A \to B$ be a mapping. Then

1) The set A is called the **domain** of f and is denoted by D_f

2) The set B is called the **Codomain** of f and is denoted by Cod_f

3) If $f(x) = y$ then y is called the **image** of .

4) The set of all images of the elements of A is called the **range** of .

$$R_f = f(A) = \{y = f(x): x \in A\}$$

ملاحظة : يمكن ايجاد المجال للدالة بالاعتماد على نوعها

1. الدالة الخطية مجالها جميع االعداد الحقيقية $D_f = R$
2. الدالة الكسرية مجالها جميع العداد الحقيقية ماعدا القيم التي تجعل المقام يساوي صفر
3. الدالة الجذرية مجالها جميع الاعداد الحقيقية عدا القيم التي تجعل القيمة تحت الجذر سالبة

لإيجاد مجموعة الصور R_f هناك عدة طرق منها الاعتماد على منحني الدالة. اذا كان مجال الدالة مجموعة جزئية من مجموعة العداد الحقيقية فأن من الممكن ايجاد المدى عن طريق ايجاد قيم x بدلالة y

4.10 Example: Write the graph set, the domain and the range of the following functions:

1) Let $f:\{-2,-1,0,1,2\} \to Z$ s.t. $f(x) = x^3$

$$Graph\ f = \{(x, x^3): x \in \{-2,-1,0,1,2\}\ and\ f(x) = x^3\}$$
$$= \{(-2,-8),(-1,-1),(0,0),(1,1),(2,8)\}$$
$$D_f = \{-2,-1,0,1,2\}$$
$$R_f = \{-8,-1,0,1,8\} \subseteq Z = Cod_f$$

2) Let $g: Z \to Z$ s.t. $g(x) = x^2$

$$Graph\ g = \{(x, x^2): x \in Z\ and\ g(x) = x^2\}$$
$$= \{\dots.,(-2,4),(-1,1),(0,0),(1,1),(2,4),\dots.\}$$
$$D_g = Z$$
$$R_g = \{0,1,4,9,16,\dots.\} \subseteq Z = Cod_g$$

4.11 Example: **(H. W.)** Find the domain and the range of the following functions:

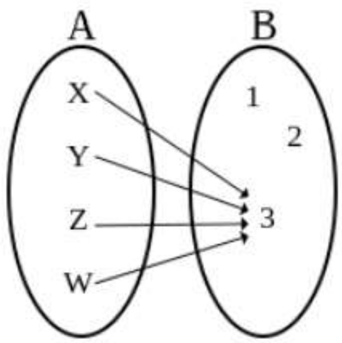
1) $f(x) = \frac{x}{x+2}$ 2) $g(x) = \sqrt{1-2x}$ 3) $h(x) = \sqrt{\frac{x}{x+2}}$

4.12 Types of Mappings انواع التطبيقات

1. Constant Mapping التطبيق الثابت

A mapping $f: A \to B$ is called Constant Mapping دالة ثابتة $\Leftrightarrow \exists!\ c \in B$ s.t $\mathrm{f(x) = c}\ \forall \mathrm{x} \in A$

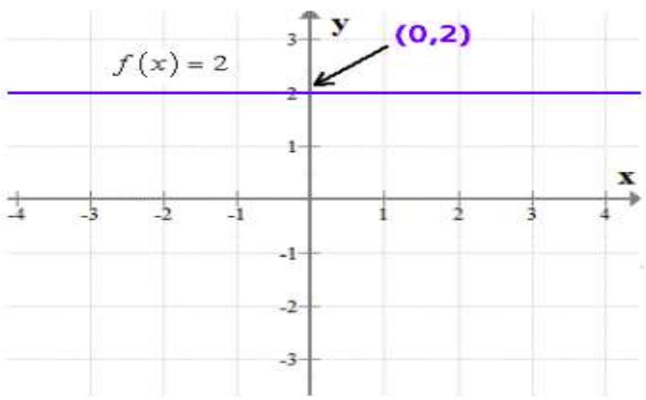
Or f is called constant $\Leftrightarrow R_f = \{c\}$



Constant Mapping

4.13 Example: Let $f: R \to R$ s.t. $f(x) = 2\ \forall x \in R$

f is constant function



4.14Example: Give two examples of non-constant functions

1. let $f: R \to R$ s.t. $f(x) = \begin{cases} 2 & , x \geq 1 \\ -3 & , x < 1 \end{cases}$

f is not a constant mapping because $f(1) = 2$ and $f(0) = -3$

Give another example (H. W.)

2. Identity Mapping الدالة الذاتية

A mapping $f: A \to A$ is called identity map دالة ذاتية denoted by $i_A \Leftrightarrow$
$f(x) = x \ \forall x \in A$

4.15 Example:

1. Let $f = \{(1,1), (3,3), (0,0), (-6,-6)\}$ f is identity function defined on $A = \{0,1,3,-6\}$ $\quad \therefore f = i_A$
2. Let $f: Z \to Z$ s.t. $f(x) = x \ \forall x \in Z \ \therefore \ f = i_Z$
3. Let $f: Z \to Z$ s.t. $f(x) = |x| \ \forall x \in Z$
   $f(x) = f(-x) = x \ \forall x \in Z \ \therefore f$ is not identity function
4. let $f: N \to N$ s.t. $f(x) = |x| \ \forall x \in N$
   $f(x) = x \ \forall x \in N \ \therefore f$ is identity function (i.e., $f = i_N$)

3. Injective Mapping التطبيق المتباين

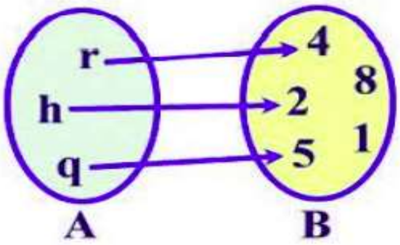
A function $f: A \to B$ is called **one to one** (1-1) or injective if different elements in the domain A have different images in B.

تسمى الدالة متباينة اذا كانت للعناصر المختلفة في المجال صورا مختلفة في المجال المقابل

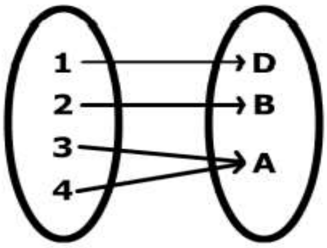
$f: A \to B$ is called 1-1 $\Leftrightarrow \forall \ x_1, x_2 \in A;\ if\ x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$

Or

$f: A \to B$ is called 1-1 $\Leftrightarrow \forall \ x_1, x_2 \in A;\ if\ f(x_1) = f(x_2)$ then $x_1 = x_2$
$f: A \to B$ is not 1-1 $\Leftrightarrow \exists \ x_1, x_2 \in A;\ x_1 \neq x_2 \ \wedge \ f(x_1) = f(x_2)$



One to one function

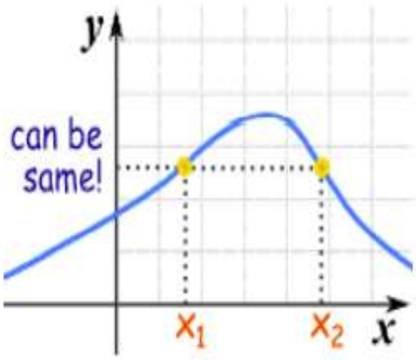


not one to one function

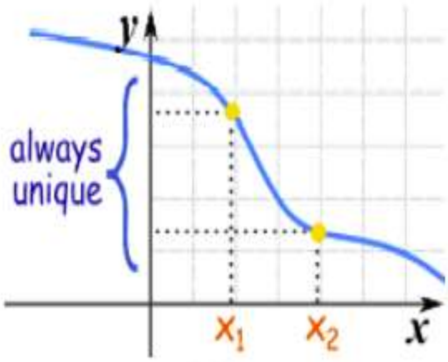
$f(3) = f(4) = A$

f is Many to one

4.16 Remark: In the graph of Injective map, a horizontal line should never intersect the curve at 2 or more points.



not one to one (many to one)



Injective (one to one) function

4.Surjective Mapping التطبيق الشامل

A function $f: A \rightarrow B$ is called (onto) or surjective if every element in "B" has at least one relating element in "A" (maybe more than one).

الدالة f تكون شاملة اذا كان كل عنصر في المجال المقابل هو صورة لعنصر واحد او اكثر في المجال.

Mathematically,

A function $f: A \rightarrow B$ is called (onto) or surjective $\Leftrightarrow R_f = Cod_f$

Or

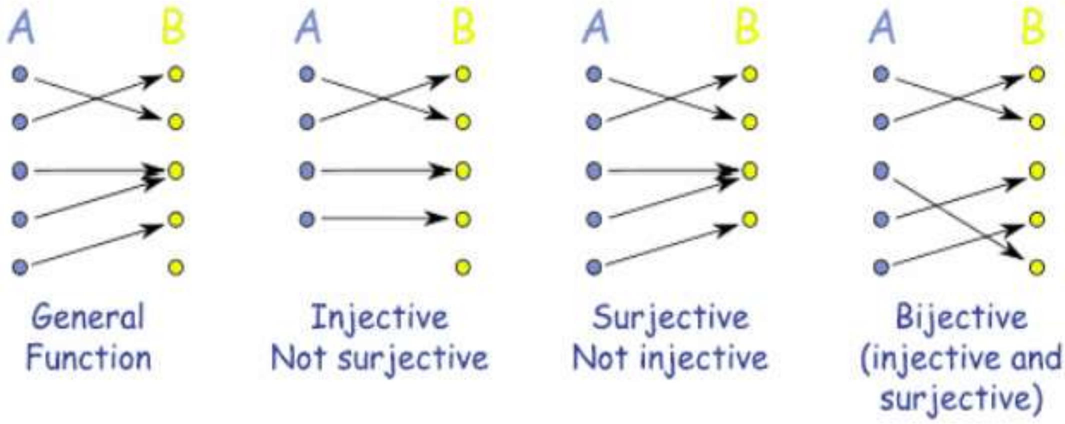
A function $f: A \rightarrow B$ is called (onto) or surjective $\Leftrightarrow \forall y \in B \; \exists x \in A \; s.t. \; f(x) = y$

A function $f: A \rightarrow B$ is not (onto) or not surjective $\Leftrightarrow R_f \neq Cod_f$

5. **bijective Mapping التطبيق المتقابل**

A function $f: A \rightarrow B$ is called bijective $\Leftrightarrow f$ is 1-1 and onto

A function $f: A \rightarrow B$ is called not bijective $\Leftrightarrow f$ is not 1-1 or f is not onto



4.17 Examples: Which of the following functions are injective? Surjective? Bijective?

1. $f: R \rightarrow R\ s.t.\ f(x) = x$ (H. W.)

2. $f: R \rightarrow R\ s.t.\ f(x) = 3$ (H. W.)

3. $f: R^+ \rightarrow R\ s.t.\ f(x) = x^2$ (H. W.)

4. $f: R \rightarrow R\ s.t.\ f(x) = \sqrt{x^2 + 9}$

5. $f: R \setminus \{\frac{5}{2}\} \rightarrow R\ s.t.\ f(x) = \frac{x+4}{2x-5}$ (H. W.)

6. $f: R \rightarrow [1, \infty)\ s.t.\ f(x) = |x - 4| + 1$

7. $f: R \rightarrow [-4, \infty)\ s.t.\ f(x) = -4 + (x - 4)^2$ (H. W.)

Solution4: $f: R \rightarrow R$ s.t. $f(x) = \sqrt{x^2 + 9}$

$D_f = R\ and\ cod_f = R$

Surjective? We need to find R_f

When $x \in R \Rightarrow f(x) = y \geq 3$

$R_f = \{y : y \geq 3\} = [3, \infty) \neq R = codf$

$\therefore f$ is not surjective (not onto).

Injective? Let $f(x_1) = f(x_2) \to x_1 = x_2$?

$f(x_1) = f(x_2) \Longrightarrow \sqrt{x_1{}^2 + 9} = \sqrt{x_2{}^2 + 9}$

$\Longrightarrow x_1{}^2 + 9 = x_2{}^2 + 9 \Longrightarrow x_1{}^2 = x_2{}^2$

$\Longrightarrow x_1 = \mp x_2 \Longrightarrow x_1 = x_2 \ldots.(1)$

$Or, x_1 = -x_2 \Longrightarrow x_1 \neq x_2 \ldots(2)$

From (2), f is not injective $\therefore f$ is not bijective

Solution6: $f: R \to [1, \infty)$ s.t. $f(x) = |x - 4| + 1 = \begin{cases} x - 3, & x \geq 4 \\ -x + 5, & x < 4 \end{cases}$

$D_f = R$ and $cod_f = [1, \infty)$

Injective? Let $f(x_1) = f(x_2) \to x_1 = x_2$?

$f(x_1) = f(x_2) \Longrightarrow |x_1 - 4| + 1 = |x_2 - 4| + 1$

$\Longrightarrow |x_1 - 4| = |x_2 - 4|$

Either $x_1 - 4 = x_2 - 4 \Longrightarrow x_1 = x_2 \ldots (1)$

$Or, x_1 - 4 = -x_2 + 4 \Longrightarrow x_1 \neq x_2 \ldots (2)$

From (2), f is not injective (1-1)

Surjective? We need to find R_f

$$f(x) = \begin{cases} x - 3, & x \geq 4 \\ -x + 5, & x < 4 \end{cases}$$

If $x \geq 4 \Longrightarrow x - 3 \geq 1 \Longrightarrow y \geq 1 \ \ldots\ldots(1)$

If $x < 4 \Longrightarrow -x > -4 \Longrightarrow 5 - x > 1 \Longrightarrow y > 1 \ \ldots\ldots.(2)$

$\therefore$ From (1) and (2), $R_f = \{y : y \geq 1 \ or \ y > 1\} = [1, \infty) = cod_f$

$\therefore f$ is surjective (onto)

طريقة اخرى لاختبار ان الدالة شاملة وهي ايجاد x بدلالة y

If $x \geq 4 \Rightarrow y = x - 3 \Rightarrow x = y + 3$

لكي تكون $4 \geq x$ يجب ان تكون قيم y اكبر او تساوي ال 1

$\Rightarrow y \geq 1 \ldots \ldots (1)$

لكي تكون $4 < x$ يجب ان تكون قيم y اكبر او تساوي ال 1

$\Rightarrow y > 1 \ldots \ldots (2)$

$\therefore$ From (1) and (2), $R_f = \{y : y \geq 1 \; or \; y > 1\} = [1, \infty) = cod_f$

$\therefore f$ is onto $\therefore f$ is not bijective

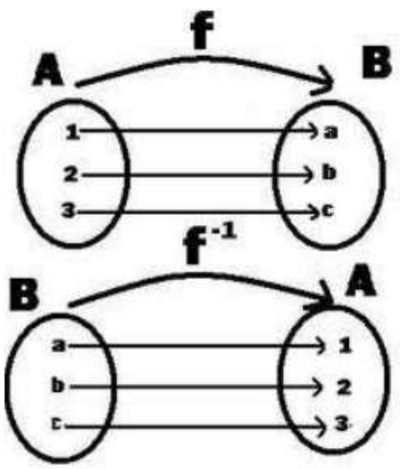
4.18 Inverse mapping: الدالة العكسية

Let f be a bijective mapping from A to B then f^{-1} is a mapping from B to A such that $f^{-1}(y) = x$.

Mathematically,

$f: A \to B$ s.t. $(x) = y$ is a bijective mapping $\Leftrightarrow f^{-1}: B \to A$ is a map. s.t. $f^{-1}(y) = x$

4.19 Example: Let $f:\{1,2,3\} \to \{a, b, c\}$ s.t. $f(1) = a,\ f(2) = b,\ f(3) = c$ Or $f = \{(1, a), (2, b), (3, c)\}$



Since f is 1-1 and onto (bijective) $\Rightarrow f^{-1}: \{a, b, c\} \to \{1,2,3\}$ s.t.

$f^{-1}(a) = 1$, $f^{-1}(b) = 2$, $f^{-1}(c) = 3$ or $f^{-1} = \{(a, 1), (b, 2), (c, 3)\}$

4.20 Example: Let $f: R^+ \to R^+$ s.t. $f(x) = \frac{x}{2}$. Find f^{-1} (if exist)?

Solution: f^{-1}exist ⟺ f is bijective

1-1? Let $f(x_1) = f(x_2) \Longrightarrow \frac{x_1}{2} = \frac{x_2}{2} \Longrightarrow x_1 = x_2 \Longrightarrow f$ is 1-1

Onto? $y = \frac{x}{2} \Longrightarrow x = 2y \Longrightarrow R_f = R^+ = cod_f$

$\therefore f$ is bijective $\Longrightarrow f^{-1}$ is a mapping $\Longrightarrow f^{-1}$ exist

لايجاد f^{-1} نجد x بدلالة y

$y = \frac{x}{2} \Longrightarrow x = 2y$

$\therefore f^{-1} : R \to R$ s.t. $f^{-1}(y) = 2y$

4.21Remark :

1. If a function f is **not injective** then f^{-1} is **not a mapping** (f^{-1}does not exist)

2. If a function f is **not surjective** then f^{-1} is **not a mapping** (f^{-1}does not exist)

3. If a function f is **bijective** then f^{-1} is a **mapping** (f^{-1} exist)

4.22Example : $f: R \to R$ s.t. $f(x) = 4 + (x - 4)^2$. Find f^{-1} (if exist)

Solution: f^{-1} exist if and only if f is 1-1 and onto

1-1? Let $(x_1) = f(x_2) \Longrightarrow 4 + (x_1 - 4)^2 = 4 + (x_2 - 4)^2$

$$\Longrightarrow (x_1 - 4)^2 = (x_2 - 4)^2$$

بجذر الطرفين

$$\Longrightarrow x_1 - 4 = \mp (x_2 - 4)$$

Either, $x_1 - 4 = x_2 - 4 \Longrightarrow x_1 = x_2$

Or, $x_1 - 4 = -(x_2 - 4) \Longrightarrow x_1 \neq x_2$

$\therefore f$ is not 1-1

$\therefore f\ -1$ is not defined

4.23 Example: $f: R \to R$ s.t. $f(x) = \begin{cases} x^3 & ,x < 0 \\ x^2 & ,x \geq 0 \end{cases}$

Find f^{-1} (if exist)?

Solution: f is bijective (see Example)

$\therefore f^{-1}$ is defined

$\therefore f^{-1} : R \to R$ s.t. $f^{-1}(y) = \begin{cases} \sqrt[3]{y} & , y < 0 \\ \sqrt{y}, & y \geq 0 \end{cases}$

4.24 Example: $f: [3, \infty) \to [0, \infty)$ s.t. $f(x) = \sqrt{x-3}$. Find f^{-1} (if exist)

Let $f(x_1) = f(x_2) \Rightarrow x_1 - 3 = x_2 - 3 \Rightarrow x_1 = x_2 \Rightarrow$ f is 1-1

Onto? Is $R_f = cod_f$?

نجد x بدلالة y

(because x ≥ 3) $y = \sqrt{x-3} \Rightarrow x = y^2 + 3 \Rightarrow y \geq 0$

يمكن ايجاد المدى بطريقة اخرى

$x \geq 3 \Rightarrow x - 3 \geq 0 \Rightarrow \sqrt{x-3} \geq 0 \Rightarrow y \geq 0$

$\Rightarrow R_f = [0, \infty) = cod_f$

$\therefore f$ is bijective

$\therefore f^{-1}$ is defined

لايجاد f^{-1} نجد x بدلالة y

$y = \sqrt{x-3} \Rightarrow x = y^2 + 3$

$\therefore f^{-1} : [0, \infty) \to [3, \infty)\ s.t.\ f^{-1}(y) = y^2 + 3$

4.25 Remark:

1. $f^{-1} \neq \frac{1}{f}$ 2. $(f^{-1})^{-1} = f$ 3. $f = f^{-1} \Leftrightarrow f = i_A$

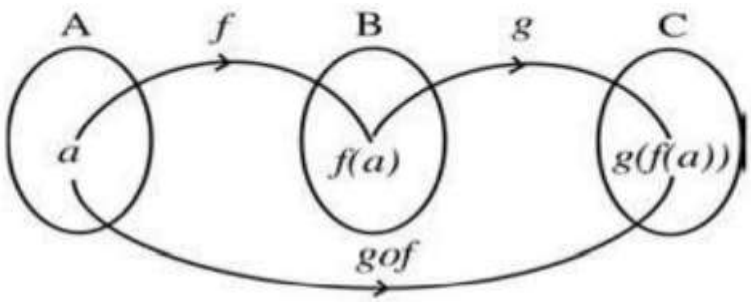
4.26 Composition of Mappings تركيب التطبيقات

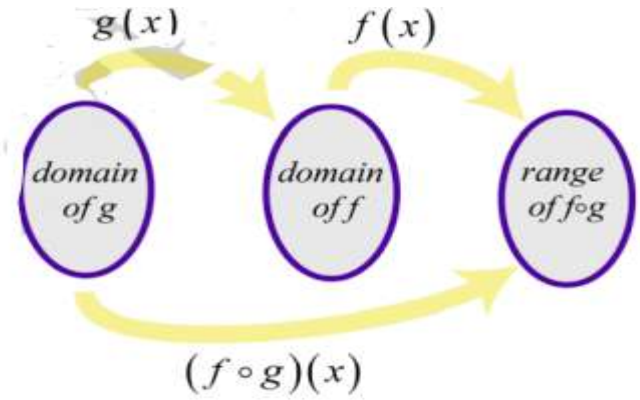
Let $f: A \to B$ be a mapping and $g: B \to C$ be a mapping. The composition of g and f is a mapping denoted by gof and is defined as

$$(gof)(a) = g(f(a)) \ \forall a \in A$$

Mathematically,

If $f: A \to B$ and g:B → C then $gof: A \to C$ is a map. ⟺ $\forall a \in A$, $\exists! \ g(f(a)) \in C \ s.t. (gof)(a) = g(f(a))$





4.27 Remark: Let $f: A \to B$ and $g: C \to D$. Then

1. gof is defined (exist) if and only if $R_f \subseteq D_g$

2. fog is defined (exist) if and only if $R_g \subseteq D_f$

3. $fog \neq gof$ (in general)

4.28 Example: Let $f: R \to [2, \infty) \ s.t. \ f(x) = x^4 + x^2 + 2$

$g: R^+ \to [-4, \infty) \ s.t. \ g(x) = \sqrt{x} - 4$. Find fog and gof (if exist)

Solution: fog exist when $R_g \subseteq D_f$

$R_g = [-4, \infty)$ and $D_f = R \Longrightarrow R_g \subseteq D_f$

$\therefore fog$ is defined

$(fog)(x) = f(g(x)) = f(\sqrt{x} - 4) = (\sqrt{x} - 4)^4 + (\sqrt{x} - 4)^2 + 2$

gof is defined when $R_f \subseteq D_g$

$R_f = [2, \infty)$ and $D_g = R^+ \Longrightarrow R_f \subseteq D_g$ $\therefore gof$ is defined

$(gof)(x) = g(f(x)) = g(x^4 + x^2 + 2) = \sqrt{x^4 + x^2 + 2} - 4$

4.29 Example: Let $f: R \to R$ $s.t.$ $f(x) = sin(x)$

$g: [0, \infty) \to (-\infty, 0]$ $s.t.$ $g(x) = -\sqrt{x}$. Find fog and gof (if exist)

Solution: fog exist when $R_g \subseteq D_f$

Find R_g? $g(x) = -\sqrt{x}$

$\sqrt{x} \geq 0 \Longrightarrow g(x) = -\sqrt{x} \leq 0$

$\therefore R_g = (-\infty, 0] \subseteq D_f = R$

$\therefore fog$ is defined

$$(fog)(x) = f(g(x)) = f(-\sqrt{x}) = sin(-\sqrt{x})$$

To find gof, we need to check if $R_f \subseteq D_g$

$(x) = \sin(x) \in [-1,1]$

$\therefore R_f = [-1,1] \subseteq D_g = [0, \infty)$ $\therefore gof$ is not a map (does not exist)

4.30 Theorem: Let $f: A \to B$ and $g: B \to C$ be two mappings, then

1. If f and g are 1-1 then gof is 1-1

2. If gof is 1-1 then f is 1-1

3. If f and g are onto then gof is onto

4. If gof is onto then g is onto

Proof 1: T.P. $gof: A \rightarrow C$ is 1-1 (injective)

$\forall x_1, x_2 \in A,\ (gof)(x_1) = (gof)(x_2)$ T.P. $x_1 = x_2$

$(gof)(x_1) = (gof)(x_2) \Rightarrow g(f(x_1)) = g(f(x_2))$ (def. of gof)

$\Rightarrow f(x_1) = f(x_2)$ (g is 1-1)

$\Rightarrow x_1 = x_2$ (f is 1-1)

$\therefore gof$ is 1-1

Proof 2: Let gof is 1-1 T.P. f is 1-1

$\forall x_1, x2 \in A, f(x_1) = f(x_2)$ T. P.$x_1 = x_2$

$f(x_1), f(x_2) \in Dg$ and $f(x_1) = f(x_2)$

$\Rightarrow g(f(x_1)) = g(f(x_2))$ (is well defined)

$\Rightarrow (gof)(x_1) = (gof)(x_2)$ (def. of gof)

$\Rightarrow x_1 = x_2$ (gof is 1-1) $\Rightarrow f$ is 1-1

Proof 3: Assume that f and g are onto T.P. gof is onto

T.P. $\forall z \in C,\ \exists x \in A\ s.t.\ (gof)(x) = z$

Since f is onto $\Rightarrow \forall y \in B,\ \exists x \in A\ s.t.\ f(x) = y$……(1)

Since g is onto $\Rightarrow \forall z \in C, \exists y \in B\ s.t.\ g(y) = z$……(2)

Substitute (1) in (2)

$\Rightarrow \forall z \in C, \exists y = f(x) \in B\ s.t.\ g(f(x)) = z$

$\Rightarrow \forall z \in C, \exists x \in A\ s.t.\ (gof)(x) = z$

$\Rightarrow gof$ is onto

Proof 4: Let gof is onto T.P. g is onto

T.P. $\forall z \in C, \exists y \in B\ s.t.\ g(y) = z$

Since gof is onto $\Rightarrow \forall z \in C, \exists x \in A\ s.t.\ (gof)(x) = z$

$\Rightarrow \forall z \in C, \exists y = f(x) \in B\ s.t.\ g(f(x)) = z$

$\Rightarrow \forall z \in C, \exists y \in B\ s.t.\ g(y) = z \Rightarrow g$ is onto

4.31 Equal mappings تساوي التطبيقات

Let f and g are two mappings, then

$$\mathrm{f} = \mathrm{g} \Leftrightarrow \mathrm{D_f} = \mathrm{D_g} \wedge \mathrm{f(x)} = \mathrm{g(x)} \forall \mathrm{x} \in \mathrm{D_f} = \mathrm{D_g}$$

4.32 Theorem : Let $f: A \to A$ be a mapping and $i_A: A \to A$ be the identity mapping, then $i_A o f = f$ and $f o i_A = f$.

Proof: T.P. $i_A o f = f \wedge f o i_A = f$

$i_A o f: A \to A$ is defined

T.P. $i_A o f = f$ لكي نبرهن المساواة يجب تحقق شرطين

1) $D_{i_A o f} = D_f$?
$i_A o f: A \to A$
$\therefore D_{i_A o f} = A = D_f$
2) $(i_A o f)(x) = f(x)$
$(i_A o f)(x) = i_A(f(x))$ (def. of o)
$= f(x)$ (def. of i_A)
$\therefore (i_A o f)(x) = f(x) \, \forall x \in A$
From (1) and (2) $\Longrightarrow i_A o f = f$
Similarly, prove that $f o i_A = f$ (H.W.)

4.33 Theorem : Let A, B and C are non empty sets. Then:
1.If $f: A \to B$ is bijective and $f^{-1}: B \to A$ then
a) $f^{-1} o f = i_A$, b) $f o f^{-1} = i_B$
2. If $f: A \to A$ is bijective and $f^{-1}: A \to A$ then
$f^{-1} o f = f o f^{-1} = i_A$ (H.W.)
3. If $f: A \to B$ be a bijective mapping then $f^{-1}: B \to A$ is a bijective map.
4. If $f: A \to B, g: B \to C, h: C \to D$ then $(hog)of = ho(gof)$
5. If $f: A \to B$ is bij. and $g: B \to C$ is bij. Then $(gof)^{-1} = f^{-1} \, o g^{-1}$

Proof 1: Let $f: A \to B$ is bijective and $f^{-1}: B \to A$
T.P. $f^{-1} o f = i_A$
$f^{-1} o f: A \to A$ is defined

1)T.P. $D_{f^{-1}of} = D_{i_A}$

$f^{-1}of: A \to A \Longrightarrow D_{f^{-1}of} = A$

$i_A: A \to A \Longrightarrow D_{i_A} = A$

$\therefore D_{f^{-1}of} = D_{i_A}$

2)T.P. $(f^{-1}of)(x) = i_A(x) \quad \forall x \in A$

$(f^{-1}of)(x) = f^{-1}\big(f(x)\big) \ldots\ldots(1)$

Since f is a map. $\Longrightarrow \forall x \in A, \exists y \in B\ s.t.\ f(x) = y \ldots\ldots(2)$

Substitute (2) in (1) $\Longrightarrow (f^{-1}of)(x) = f^{-1}(y) = x$

$\therefore (f^{-1}of)(x) = x = i_A(x)$

From (1) and (2) $\Longrightarrow .\ f^{-1}of = i_A$

Next we prove $fof^{-1} = i_B$

$fof^{-1} : B \to B$ is defined

1)T.P. $D_{fof^{-1}} = D_{i_B}$

$fof^{-1} : B \to B \Longrightarrow D_{fof^{-1}} = B$

$i_B: B \to B \Longrightarrow D_{i_B} = B$

$\therefore D_{fof^{-1}} = D_{i_B}$

2)T.P. $(fof^{-1})(y) = i_B(y)\ \forall y \in B$

$(fof^{-1})(y) = f(f^{-1}(y)) \ldots \ldots .(1)$

Since f^{-1} is a map. $\Longrightarrow \forall y \in B, \exists x \in A\ s.t.\ f^{-1}(y) = x \ldots\ldots(2)$

Substitute (2) in (1) $\Longrightarrow (fof^{-1})(y) = f(x) = y$

$\therefore (f^{-1}of)(y) = y = i_B(y)$

From (1) and (2) $\Longrightarrow f^{-1}of = i_B$

Proof 3: Let $f: A \to B$ be a bijective mapping (1-1 and onto)

T.P. $f^{-1}: B \to A$ is a bijective mapping.

$f^{-1}$is 1-1? Let $y_1, y_2 \in B$ s.t. $f^{-1}(y_1) = f^{-1}(y_2)$ T.P. $y_1 = y_2$

since $y_1, y_2 \in B = R_f \Longrightarrow \exists x_1, x_2 \in A\ s.t.$

$f(x_1) = y_1, f(x_2) = y_2$ [f is onto]

$\Longrightarrow \exists x_1, x_2 \in A$ s.t.

$f^{-1}(f(x_1)) = f^{-1}(y_1)$ and $f^{-1}(f(x_2)) = f^{-1}(y_2)$ [f^{-1} is well defined]

$\Longrightarrow \exists x_1, x_2 \in A$ s.t. $f^{-1}(f(x_1)) = f^{-1}(f(x_2)$ [$f^{-1}(y_1) = f^{-1}(y_2)$]
$\Longrightarrow \exists x_1, x_2 \in A$ s.t. $(f^{-1}of)(x_1) = (f^{-1}of)(x_2)$ [def. of $f^{-1}of$]

$\Longrightarrow \exists x_1, x_2 \in A$ s.t. $i_A(x_1) = i_A(x_2)$ [from part (1)]

$\Longrightarrow x_1 = x_2 \quad \Longrightarrow f(x_1) = f(x_2)$ [f is well defined]

$\Longrightarrow y_1 = y_2 \quad \therefore f^{-1}$ is 1-1

f^{-1} is onto? T.P. $R_{f^{-1}} = A$

$R_{f^{-1}} = \{x \in A: x = f - 1\ (y)\} = \{x \in A: f(x) = y\} = A$

$\therefore f^{-1}$ is onto

$f^{-1}$is 1-1 and onto $\Longrightarrow f^{-1}$ is bijective

Proof 4: let $f: A \to B$, g: B → C, $h: C \to D$ T.P. $(hog)of = ho(gof)$

1) $D(hog)of = Dho(gof) = A$

2) T.P. $\forall x \in A, [(hog)of](x) = [ho(gof)](x)$

$[(hog)of](x) = (hog)(f(x)$ (def. of o)

$= h(g(f(x))$ (def. of o)

$= h(gof(x)) = [ho(gof)](x)$

$\therefore (hog)of = ho(gof)$

Proof 5: Let $f: A \rightarrow B$ is bijective and $g: B \rightarrow C$ is bijective

T. P. $(gof)^{-1} = f^{-1} o g^{-1}$

Let $h = gof: A \rightarrow C$ T.P. $h^{-1} = f^{-1} o g^{-1}$

T.P. $hoh^{-1} = i_C$

$hoh^{-1} = (gof)o(f^{-1} o g^{-1})$

$= go(fof^{-1})og^{-1}$ [o is associative]

$= goi_B og^{-1}$ $[fof^{-1} = i_B]$

$= gog^{-1}$ $[goi_B = g]$

$= i_C$ $[gog^{-1} = i_C]$

$\therefore (gof)^{-1} = f^{-1} o g^{-1}$

4.34 Direct Image الصورة المباشرة

Let $f: A \rightarrow B$ be a mapping and $C \subseteq A$. Then the direct image of C under f is defined as

$$f(C) = \{y \in B; \exists x \in C\ s.t. y = f(x)\}$$

الصورة المباشرة $f(C)$ هي مجموعة جزئية من المجال المقابل B والتي كل عنصر فيها هو صورة لعنصر او اكثر من عناصر المجموعة C الجزئية من المجال. وتسمى $f(C)$ الصورة المباشرة ل C بفعل التطبيق f.

$$y \in f(C) \Leftrightarrow \exists x \in C\ s.t. y = f(x)$$

$$y \notin f(C) \Leftrightarrow \forall x \in C\ s.t. y \neq f(x)$$

4.35 Example: Let $f: N\backslash\{1\} \rightarrow N\ s.t. f(n) = n^2 - 1$

Let $C = \{2,3,4\}$. Find $f(C)$

Solution:

$f(C) = \{f(2), f(3), f(4)\} = \{3,8,15\}$

<u>4.36 Theorem:</u> Let $f: A \rightarrow B$ be a mapping, let $C and\ D$ are subsets of A. Then:

1. If $C \subseteq D$ then $f(C) \subseteq f(D)$

2. $f(C \cap D) \subseteq f(C) \cap f(D)$ (H.W.)

3. If f is injective (1-1) then $f(C \cap D) = f(C) \cap f(D)$

4. $f(C \cup D) = f(C) \cup f(D)$

5. $f(C)\backslash f(D) \subseteq f(C\backslash D)$

6. $f(C\backslash D) \subseteq f(C)$ (H.W.)

Proof1: Let $C \subseteq D$ T.P. $f(C) \subseteq f(D)$

Let $y \in f(C) \Rightarrow y \in B; \exists x \in C\ s.t.\ y = f(x)$ (def. of $f(C)$)

$\Rightarrow y \in B; \exists x \in D\ s.t.\ y = f(x) \quad (C \subseteq D)$

$\Rightarrow y \in f(D)$ (def. of $f(D)$)

$\therefore f(C) \subseteq f(D)$

Proof3: Suppose f is 1-1

T.P. $f(C \cap D) \subseteq f(C) \cap f(D) \wedge f(C) \cap f(D) \subseteq f(C \cap D)$

From (2), $f(C \cap D) \subseteq f(C) \cap f(D)$ ….(1) (H.W.)

T.P. $f(C) \cap f(D) \subseteq f(C \cap D)$

Let $y \in f(C) \cap f(D)$ T.P. $y \in f(C \cap D)$

$y \in f(C) \cap f(D) \Rightarrow y \in f(C) \wedge y \in f(D)$ (def. of $\cap$)

$\Rightarrow \exists\, x_1 \in C, y = f(x_1) \wedge \exists\, x_2 \in D,\ y = f(x_2)$ (def. of direct image)

$\Rightarrow y = f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ (f is 1-1)

$\Rightarrow \exists\, x = x_1 = x_2 \in C \cap D$ s.t. $y = (x) \in (C \cap D)$ (def. of direct image)

$\therefore f(C) \cap f(D) \subseteq f(C \cap D)$ ….(2)

From (1) and (2), $f(C) \cap f(D) = f(C \cap D)$

Proof4: T.P. $f(C \cup D) \subseteq f(C) \cup f(D) \ \wedge f(C) \cup f(D) \subseteq f(C \cup D)$

Let $y \in f(C \cup D) \Leftrightarrow y \in B;\ \exists x \in C \cup D\ s.t.\ y = f(x)$ (def. of $f(C \cup D)$)

$\Leftrightarrow y \in B;\ \exists x \in C \vee x \in D\ s.t.\ y = f(x)$ (def. of ∪)

$\Leftrightarrow y \in B; [\exists x \in C\ s.t.\ y = f(x)] \vee [x \in D\ s.t.\ y = f(x)]$

$\Leftrightarrow y \in f(C) \vee y \in f(D)$ (def. of direct image)

$\Leftrightarrow y \in f(C) \cup f(D)$ (def. of ∪)

$\therefore f(C \cup D) = f(C) \cup f(D)$

Proof5: T.P. $f(C)\backslash f(D) \subseteq f(C\backslash D)$

Let $y \in f(C)\backslash f(D) \Rightarrow y \in f(C) \wedge y \notin f(D)$ $(def. of \backslash)$

$\Rightarrow \exists x \in C\ s.t.\ y = f(x) \wedge \forall x \in D;\ y \neq f(x)$ (def. of direct image)

$\Rightarrow \exists x \in C\ s.t.\ y = f(x) \wedge x \notin D;\ y = f(x)$

$\Rightarrow \exists x \in C \wedge x \notin D;\ y = f(x)$

$\Rightarrow y \in f(C\backslash D)$

$\therefore f(C)\backslash f(D) \subseteq f(C\backslash D)$

4.37 Inverse Image الصورة العكسية

Let $f: A \rightarrow B$ be a mapping and $D \subseteq B$. Then the inverse image of D under f is defined as

$$f^{-1}(D) = \{x \in A: f(x) \in D\}$$

الصورة العكسية $f^{-1}(D)$ هي مجموعة جزئية من المجال A والتي تنتمي صورة كل عنصر فيها الى المجموعة D الجزئية من المجال. وتسمى $f^{-1}(D)$ الصورة العكسية ل D بفعل التطبيق f .

$$x \in f^{-1}(D) \Leftrightarrow x \in A\ s.t.\ f(x) \in D$$

$$x \notin f^{-1}(D) \Leftrightarrow x \in A\ s.t.\ f(x) \notin D$$

<u>4.38 Example</u>: Let $f: N\backslash\{1\} \rightarrow N \ s.t. \ f(n) = n^2 - 1$

Let $D = \{2,3,4,8\}$. Find $f^{-1}(D)$

Solution: $f^{-1}(D) = \{n \in N\backslash\{1\}: f(n) \in \{2,3,4,8\}\}$

$= \{n \in N\backslash\{1\}: n^2 - 1 = 2 \lor n^2 - 1 = 3 \lor n^2 - 1 = 4 \lor n^2 - 1 = 8\}$

$= \{n \in N\backslash\{1\}: n^2 = 3 \lor n^2 = 4 \lor n^2 = 5 \lor n^2 = 9\}$

$= \{n \in N\backslash\{1\}: n = \sqrt{3} \notin N \lor n = 2 \in N \lor n = \sqrt{5} \notin N \lor n = 3 \in N\}$

$\therefore f^{-1}(D) = \{2,3\}$

<u>4.39 Example:</u> Let $f: R \rightarrow R \ s.t. \ f(x) = x^2 - 2$. Find $f^{-1}(\{2,7\})$

Solution: $f^{-1}(\{2,7\}) = \{x \in R: f(x) \in \{2,7\}\}$

$= \{x \in R: x^2 - 2 = 2 \ or \ x^2 - 2 = 7\}$

$= \{2, -2, 3, -3\}$

<u>4.40 Theorem:</u>

Let $f: A \rightarrow B$ be a mapping, let $E \subseteq A$, $C \ and \ D$ are subsets of B. Then:

1. $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$

2. $If \ C \subseteq D \ then \ f^{-1}(C) \subseteq f^{-1}(D)$ (H.W.)

3. $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$ (H.W.)

4. $f^{-1}(C\backslash D) = f^{-1}(C)\backslash f^{-1}(D)$

5.$f^{-1}(C\backslash D) \subseteq f^{-1}(C)$ (H.W.)

6. $E \subseteq f^{-1}(f(E))$ (H.W.)

7. If f is 1-1 if and only if $E = f^{-1}(f(E))$

8. $f(f^{-1}(C)) \subseteq C$ (H.W.)

9. If f is onto if and only if $f(f^{-1}(C)) = C$

Proof1: Let $x \in f^{-1}(C \cap D) \Leftrightarrow f(x) \in C \cap D$ $(def. of\ f^{-1})$

$\Leftrightarrow f(x) \in C \wedge f(x) \in D$ $(def. of\ \cap)$

$\Leftrightarrow x \in f^{-1}(C) \wedge x \in f^{-1}(D)$ (def. of f^{-1})

$\Leftrightarrow x \in f^{-1}(C) \cap f^{-1}(D)$ $(def. of\ \cap)$

$\therefore f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$

Proof4: Let $x \in f^{-1}(C \setminus D) \Leftrightarrow f(x) \in C \setminus D$ (def. of f^{-1})

$\Leftrightarrow f(x) \in C \wedge f(x) \notin D$ $(def. of\ \setminus)$

$\Leftrightarrow x \in f^{-1}(C) \wedge x \notin f^{-1}(D)$ (def. of f^{-1})

$\Leftrightarrow x \in f^{-1}(C) \setminus f^{-1}(D)$ $(def. of\ \setminus)$

$\therefore f^{-1}(C \setminus D) = f^{-1}(C) \setminus f^{-1}(D)$

Proof 7: $\Rightarrow$) Let f is 1-1 T.P. $E \subseteq f^{-1}(f(E)) \wedge f^{-1}(f(E)) \subseteq E$

From part (6), $E \subseteq f^{-1}(f(E))$ …….(1) T.P. $f^{-1}(f(E)) \subseteq E$

Let $x \in f^{-1}(f(E)) \Rightarrow f(x) \in f(E)$ (def. of inverse image)

$\Rightarrow \exists\ e \in E\ s.t. f(x) = f(e)$

$\Rightarrow x = e \in E$ (f is 1-1)

$\Rightarrow x \in E$

$\therefore f^{-1}(f(E)) \subseteq E$ ……(2)

From (1) and (2), $f^{-1}(f(E)) = E$

$\Leftarrow$)Assume that $f^{-1}(f(E)) = E$ T.P. f is 1-1

Suppose f is not 1-1 برهان غير مباشر

$\exists x_1, x_2 \in A\ s.t\ f(x_1) = f(x_2)$ and $x_1 \neq x_2$

Let $E = \{x_1\} \Rightarrow x_1 \in E \Rightarrow f(x_1) \in f(E)$ (def. of direct image)

$\Longrightarrow f(x_2) \in f(E) \quad (f(x_1) = f(x_2))$

$\Longrightarrow f(x_1) \in f(E)$ and $f(x_2) \in f(E)$

$\Longrightarrow x_1 \in f^{-1}(f(E)) = E$ and $x_2 \in f^{-1}(f(E)) = E$

$\Longrightarrow x_1, x_2 \in E = \{x_1\}$ تناقض

$\Longrightarrow$ is 1-1

Proof 9: $\Longrightarrow$) Let f is onto T.P. $f(f^{-1}(C)) \subseteq C \wedge C \subseteq f(f^{-1}(C))$

From part (8), $f(f^{-1}(C)) \subseteq C$ …….(1)

T.P. $C \subseteq f(f^{-1}(C))$

Let $y \in C \Longrightarrow \exists x \in A$ s.t. $y = f(x)$ (f is onto)

$\Longrightarrow x \in f^{-1}(y) \in f^{-1}(C)$ (def. of inverse image)

$\Longrightarrow f(x) = y \in f(f^{-1}(C))$

$\therefore C \subseteq f(f^{-1}(C))$ …..(2)

From (1) and (2), $C = f(f^{-1}(C))$

$\Longleftarrow$)Assume that $f(f^{-1}(C)) = C$ T.P. f is onto

Assume f is not onto برهان بالتناقض

$\exists y \in B - f(A) \Longrightarrow y \neq f(x) \, \forall x \in A$

$\Longrightarrow f^{-1}(y) \neq x \, \forall x \in A$

Let $C = \{y\} \Longrightarrow f^{-1}(C) = f^{-1}(\{y\}) = \emptyset$

$\Longrightarrow f(f^{-1}(C)) = f(\emptyset) = \emptyset$

$\Longrightarrow f(f^{-1}(C)) \neq C$ تناقض مع الفرض

$\therefore f$ is onto

4.41 Remark: Let $f: A \rightarrow B$ be a mapping, let $E \subseteq A$ and $C \subseteq B$. Then in general

1. $A \neq f^{-1}(f(A))$
2. $B \neq f(f^{-1}(B))$

For example, Let $A = \{1,2,3\}$, $B = \{4,5,6,7\}$

$f: A \rightarrow B$ s.t. $f(1) = f(2) = 4$, $f(3) = 6$

Let $E = \{1,3\} \subseteq A$ and $C = \{4,5\} \subseteq B$

$f(E) = \{4,6\} \Rightarrow f^{-1}(f(E)) = f^{-1}(\{4,6\}) = \{1,2,3\} \neq E$

$\Rightarrow f^{-1}(f(E)) \neq E$

Also, $f^{-1}(C) = \{1,2\} \Rightarrow f(f^{-1}(C)) = f(\{1,2\}) = \{4\} \neq C$

$\Rightarrow f(f^{-1}(C)) \neq C$.

Chapter Five:

Cardinality of Sets قدرة المجموعات

5.1 Equivalent of Two Sets : التكافؤ بين مجموعتين

Two sets A and B are called equivalent ($A \approx B$) if and only if there exist a bijective map between them.

المجموعتان $\boldsymbol{A}$ و $\boldsymbol{B}$ تسمى مجموعتين متقابلتين او متكافئتين اذا وجدت دالة متقابلة تربط بينهما

Mathematically,

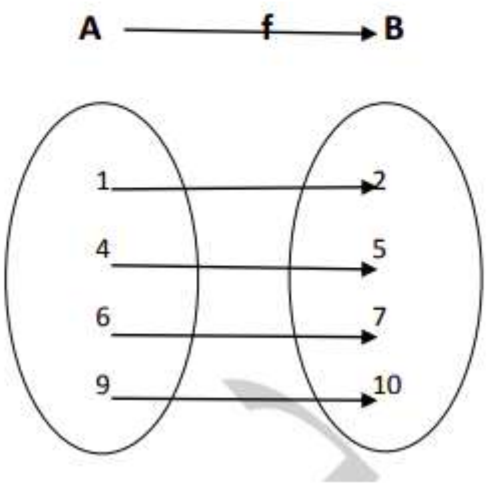
$$A \approx B \iff \exists f \; s.t. f: A \to B \; is \; bijective$$

$$A \not\approx B \iff \forall f \; s.t. f: A \to B \; is \; not \; bijective$$

5.2 Remark: If $A \approx B$, we say that there is a "**one to one correspondence**" between A and B.

5.3 Example: Let $A = \{1,4,6,9\}$, $B = \{2,5,7,10\}$. Show that $A \approx B$. Is $B \approx A$?

Solution: Define $f: A \to B \; s.t. \; f(1) = 2, f(4) = 5, f(6) = 7, f(9) = 10$
In general, $f(x) = x + 1 \quad \forall x \in A$.



1-1? $\forall x_1, x_2 \in A, \; x_1 \neq x_2 \Longrightarrow f(x_1) \neq f(x_2) \Longrightarrow f$ is 1-1

Onto? $R_f = \{2,5,7,10\} = cod_f$

$\therefore f$ is bijective

$\Longrightarrow A \approx B$ (there is a one to one correspondence between A and B)

Give another bijective function that makes $A \approx B$?

Define $h: A \rightarrow B$ $s.t.$ $h(1) = , h(4) = , h(6) = , h(9) =$

Is $B \approx A$?

Define $g: B \rightarrow A$ $s.t$ $g(2) = 1,\ g(5) = 4,\ g(7) = 6,\ g(10) = 9$

Or $g(x) = x - 1\ \forall x \in B$

g is 1-1 and onto (check)

$\therefore g$ is bijective $\Longrightarrow B \approx A$

طريقة ثانية لبرهان ان $B \approx A$

Since $A \approx B \Longrightarrow \exists f^{-1} : B \rightarrow A$ $s.t.$ $f^{-1}(y) = x$

لإيجاد قيمة x

Let $y = x + 1 \Longrightarrow x = y - 1$

Substitute the value of x in $f^{-1}(y)$

$\therefore f^{-1} : B \rightarrow A$ $s.t.$ $f -1\ (y) = y - 1$

Show that $f^{-1}$is bijective (H.W.)

$\therefore B \approx A$

5.4 Example: Let $A = \{x, y, z\}, B = \{0, -1\}$

Is $A \approx B$? Is $B \approx A$?

Solution: $A \approx B \Leftrightarrow \exists f$ $s.t.$ $f: A \rightarrow B$ is bijective

Let $f(x) = 0\ , f(y) = -1\ ,\ f(z) = -1$

This relation is not 1-1 mapping because two element $y, z \in A$ have the same image.

In fact each $f: A \rightarrow B$ is not 1-1$\Longrightarrow f$ is not bijective

$\Longrightarrow$ A is not equivalent to B.

$B \approx A \Leftrightarrow \exists g \; s.t. \; g: B \rightarrow A$ is bijective Let $g(0) = x$, $g(-1) = y$

This relation is not onto mapping because $Rg \neq codg = A$

In fact each $g: B \rightarrow A$ is not onto $\Longrightarrow g$ is not bijective

$\Longrightarrow$ B is not equivalent to A.

5.6 Example: Show that $N \approx B = \{5,10,15,20, \ldots\}$

Solution: Define $f: N \rightarrow B \; s.t. \; f(x) = 5x \; \forall x \in N$

1-1? Let $x_1, x_2 \in N \; s.t. \; f(x_1) = f(x_2) \Longrightarrow 5x_1 = 5x_2$

$\Longrightarrow x_1 = x_2$

$\Longrightarrow f$ is injective

Onto? $R_f = \{y \in B: y = 5x, x \in N\} = \{5,10,15,20, \ldots ..\} = B$

$\therefore f$ is onto $\therefore N \approx B$

5.7 Example: **(H.W.)** Show that

1) $N \approx E^+$ 2) $N \approx E^-$ 3) $N \approx Z^-$

5.8 Theorem: The equivalent relation ($\approx$) on sets is an equivalence relation.

Proof:

1.$\approx$ Reflexive? T.P. $A \approx A$ for any set A $\exists i_A: A \rightarrow A \; s.t. \; i(x) = x \; \forall x \in A$

i_A is bijective $\Longrightarrow A \approx A \; \forall A$
$\therefore \approx$ reflexive

2. $\approx$ Symmetric? Let A and B are two sets such that $A \approx B$ T.P. $B \approx A$

$A \approx B \Longrightarrow \exists f \; s.t. f: A \rightarrow B$ is bijective Since f is bij.

$\Longrightarrow \exists f^{-1} : B \rightarrow A$ such that f^{-1} is bij.

$\Longrightarrow B \approx A$

$\therefore \approx$ is symmetric

3. ≈ transitive? Let A, B and c are sets

such that $A \approx B$ and $B \approx C$ T.P. $A \approx C$

$A \approx B \Longrightarrow \exists f\ s.t. f: A \to B$ is bijective

$B \approx C \Longrightarrow \exists g\ s.t. g: B \to C$ is bijective

$\therefore \exists gof: A \to C$ is bijective. (by theorem (chapter4))

$\therefore A \approx C$

$\therefore \approx$ is transitive

$\therefore \approx$ is an equivalence relation.

5.9 Finite and infinite sets **المجموعات المنتهية وغير المنتهية**

A set A is said to be **finite** if A is empty or if A contains exactly m elements where m is a positive integer; otherwise A is **infinite**.

المجموعة A تسمى **منتهية** اذا كانت مجموعة خالية او اذا كانت تحوي على عدد منتهي m من العناصر. وفيما عدا ذلك تكون المجموعة A **غير منتهية**.

5.10 Remark: The number of the elements in a finite set A is called the **size** of $\boldsymbol{A}$ and is denoted by $n(A)$ or $\#(A)$ or $|A|$.

5.11 Examples:

Let $A = \{1,2\}$ finite set $\Longrightarrow n(A) = 2$ عدد عناصر المجموعة

Let $A = \{\ \}$ finite set $\Longrightarrow \#(A) = 0$

Let $A = \{\emptyset\}$ finite set $\Longrightarrow \#(A) = 1$

Let $A = \{1, \emptyset, \{1,3\}, [0,1], N\}$ finite set $\Longrightarrow |A| = 5$

5.12 Remark: If A is an infinite set $\Longrightarrow |A|\ or\ \#(A)$ is not defined (does not exist).

5.13 Example: Let $A = Z$ infinite set $\Longrightarrow |A|$ is not defined

Let $A = (9,40]$ infinite set $\Longrightarrow \#(A)$ does not exist .

5.14 Cardinality of Finite Sets **قدرة المجموعة المنتهية**

Let $J_m = \{1,2,\dots,m\}$ be a set. A set A is called finite of size m ($(A) = m$) if and only if $A \approx J_m = \{1,2,\dots,m\}$. The positive number m is called the cardinality of $\boldsymbol{A}$.

اذا كانت المجموعة منتهية فان عدد عناصرها يسمى قدرة المجموعة

Mathematically,

$$n(A) = m \Leftrightarrow A \approx \{1,2,\dots,m\} = Jm$$

$$n(A) = m \Leftrightarrow \exists f: A \to \{1,2,\dots,m\}\ s.t.\ f \text{ is bijective}$$

5.15 Example: Let $A = \{\alpha, \beta, \sin(x)\}$. Find the cardinality of A.

Solution: Let $J_3 = \{1,2,3\}$ T.P. $A \approx J_3$

Define $f: A \to J_3$ $s.t.$ $f(\alpha) = 1,\ f(\beta) = 2,\ f(\sin(x)) = 3$

It is clear that f is 1-1 and onto $\Rightarrow n(A) = 3$

5.16 Theorem: Any two finite sets have the same cardinal number if and only if there is a bijective map. between them

i.e., Let A and B be two finite sets. Then $n(A) = n(B) \Leftrightarrow A \approx B$

Proof: $\Rightarrow$) let $n(A) = n(B) = m$ ***T.P.*** $A \approx B$

$n(A) = m \Rightarrow A \approx \{1,2,\dots,m\}$….(1) (def. of $n(A)$)

$n(B) = m \Rightarrow B \approx \{1,2,\dots,m\}$….(2) (def. of $n(B)$)

From (1) &(2) $A \approx \{1,2,\dots,m\} \wedge B \approx \{1,2,\dots,m\}$

$\Rightarrow A \approx \{1,2,\dots,m\} \wedge \{1,2,\dots,m\} \approx B$ [$\approx$ symmetric]

$\Rightarrow A \approx B$ [$\approx$ transitive]

$\Leftarrow$) Suppose $A \approx B$ T.P. $n(A) = n(B)$

Let $n(A) = m \Rightarrow A \approx \{1,2,\dots,m\}$ (def. of (A))

$\Rightarrow A \approx B \wedge A \approx \{1,2,\dots,m\}$

$\Longrightarrow \mathrm{B} \approx \mathrm{A} \wedge \mathrm{A} \approx \{1,2,\ldots,\mathrm{m}\}$ [≈ symmetric]

$\Longrightarrow \mathrm{B} \approx \{1,2,\ldots,\mathrm{m}\}$ [≈ transitive]

$\Longrightarrow \mathrm{n(B)} = \mathrm{m}$

$\Longrightarrow \mathrm{n(A)} = \mathrm{n(B)}$

5.17 Theorem: Let A be a finite set and $\mathrm{A}_1, \mathrm{A}_2, \ldots, \mathrm{A}_\mathrm{n}$ be a partition of A. Then $n(A) = n(\mathrm{A}_1), n(\mathrm{A}_2), \ldots, \mathrm{n}(\mathrm{A}_\mathrm{n})$ (بدون برهان) ……*

5.18 Theorem: Let A and B be finite sets. Then

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

Proof: Take $A \backslash B,\ A \cap B,\ B \backslash A$ as partition for $A \cup B$

$\Longrightarrow n(A \cup B) = n(A \backslash B) + n(A \cap B) + n(B \backslash A) \ldots (1)$ (by theorem…*)

Take $A \backslash B, A \cap B$, as partition for A

$\Longrightarrow n(A) = n(A \backslash B) + n(A \cap B)$ (by theorem ….*)

$\Longrightarrow n(A \backslash B) = n(A) - n(A \cap B) \ldots (2)$

Take $B \backslash A, A \cap B$, as partition for B

$\Longrightarrow n(B) = n(B \backslash A) + n(A \cap B)$ (by theorem ….*)

$\Longrightarrow n(B \backslash A) = n(B) - n(A \cap B) \ldots (3)$

Substitute (2)&(3) into (1)

$\Longrightarrow n(A \cup B) = n(A) - n(A \cap B) + n(A \cap B) + n(B) - n(A \cap B)$

$\Longrightarrow n(A \cup B) = n(A) + n(B) - n(A \cap B)$

5.19 Theorem: Let A and B be finite sets. Then $n(A \times B) = n(A).n(B)$

Proof: A is finite set $\Longrightarrow n(A) = m$ s.t. $A = \{a_1, a_2, \ldots, a_m\}$

B is finite set $\Longrightarrow n(B) = n$ s.t. $B = \{b_1, b_2, \ldots, b_n\}$

$A \times B = \{(a,b): a \in A, b \in B\}$

$A \times \{b1\} = \{(a1, b1), (a2, b1), \dots, (am, b1)\}$

$A \times \{b2\} = \{(a1, b2), (a2, b2), \dots, (am, b2)\}$

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$A \times \{bn\} = \{(a1, bn), (a2, bn), \dots, (am, bn)\}$

$\Longrightarrow A \times B = A \times \{b1\} \cup A \times \{b2\} \cup \dots \cup A \times \{bn\}$

Such that $A \times \{bi\} \cap A \times \{bj\} = \emptyset \ \forall i \neq j$

$\therefore A \times \{b1\}, A \times \{b2\}, \dots, A \times \{bn\}$ is a partition for $A \times B$

$n(A \times B) = n(A \times \{b1\}) + n(A \times \{b2\}) + \cdots + n(A \times \{bn\})$

$$= \underbrace{m + m + \cdots + m}_{n^{-}times} = mn = n(A).\, n(B)$$

5.20 Cardinality of infinite sets قدرة المجموعات غير المنتهية

Let A be an infinite set. Then the cardinality of A is not a finite positive number. The cardinality of A is denoted by $\aleph_0$.

المجموعة غير المنتهية تكون لها قدرة ولكن القدرة هنا التكون عدد معرف موجب ترمز القدرة المجموعة الغير منتهية الى حجم تلك المجموعة

5.21 Example: The cardinal number of N is denoted by $\aleph_0$

i.e., $n(N) = \#(N) = \aleph_0$

5.22 Countably infinite set المجموعة غير المنتهية القابلة للعد

An infinite set A is called countable if it is equivalent to the set of natural number. Thus, the cardinality of an infinite countable set is $\aleph_0$

Mathematically,

A is countable infinite set $\Leftrightarrow A \approx N \Leftrightarrow n(A) = n(N) = \aleph_0$

A is not countable infinite set $\Leftrightarrow A \approx N$

5.23 Example: Show that N is countably infinite

Solution: T. P. $N \approx N$

Define $f: N \rightarrow N \; s.t. \; f(x) = i_N(x) = x \; \forall x \in N$

f is bijective (prove!)

$\therefore N \approx N$

5.24 Example: Show that the set of even positive numbers is countably infinite

Solution: T. P. $E^+ \approx N$

يمكن ايجاد دالة متقابلة بين المجموعتين اما بالتخمين او من خالل ايجاد معادلة مستقيم اذا علمت منه نقطتان

$$E^+ : \; 0 \;\; 2 \;\; 4 \;\; 6 \;\; 8 \;\; 10 \;\; \ldots$$

$$N \;\; : \; 1 \;\; 2 \;\; 3 \;\; 4 \;\; 5 \;\; 6 \;\; \ldots$$

Let $f(0) = 1$ and $f(2) = 2$

$(x_1, y_1) = (0,1)$ and $(x_2, y_2) = (2,2)$

نجد معادلة المستقيم المحدد بالنقطتين (x_1, y_1) و (x_2, y_2)

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\frac{y-1}{x} = \frac{1}{2} \Rightarrow y - 1 = \frac{x}{2} \qquad \Rightarrow y = \frac{x}{2} + 1$$

$\therefore f: E^+ \rightarrow N \; s.t. \; f(x) = \frac{x}{2} + 1$ f is bijective (prove!) $\therefore E^+ \approx N$

5.25 Example: **(H. W.)** Prove that

1) $A = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ..\}$ is countably infinite set.

2) $A = Z$ is countable infinite set .

3) $A_k = \{0, \mp k, \mp 2k, \mp 3k, \ldots\}$ is countably infinite set.

4) $A = O^+$ is countable infinite set.

5.26 Theorem: Any infinite subset of an infinite countable set is countable.

i.e., If A is countable infinite set and $B \subseteq A$ then B is countable set.

اي مجموعة جزئية من مجموعة قابلة للعد تكون قابلة للعد.

5.27 Theorem: If A is countably infinite set then $A \cup \{a\}$ is also countably infinite set.

Proof: Let A be a countably infinite set $\Longrightarrow A \approx N$

$\exists f: A \to N$ bijective $s.t.\ f(a_1) = 1, f(a_2) = 2, f(a_3) = 3, \ldots.$

$A \quad a_1 \quad a_2 \quad a_3 \ \ldots$

$N \quad 1 \quad 2 \quad 3 \ \ldots$

T.P. $A \cup \{a\} \approx N$

Define $g: A \cup \{a\} \to N$ s.t.

$$g(a) = 1, g(a_1) = 2, g(a_2) = 3, g(a_3) = 4, \ldots.$$

$$A \cup \{a\}: \quad a \quad a_1 \quad a_2 \quad a_3 \quad \ldots..$$

$$N \quad : \quad 1 \quad 2 \quad 3 \quad 4 \ldots.$$

$g(a) = 1$

$g(a_1) = 2 = 1 + 1 = f(a_1) + 1$

$g(a_2) = 3 = 2 + 1 = f(a_2) + 1$

$g(a_3) = 4 = 3 + 1 = f(a_3) + 1$

In general,

$$g(x) = \begin{cases} 1 \quad , & x = a \\ f(x) + 1 \ , & x \neq a \end{cases}$$

T.P. g is bijective

***g* is 1-1?** Let $x_1, x_2 \in A \cup \{a\}\ s.t.\ g(x_1) = g(x_2)$ T.P. $x_1 = x_2$

$$g(x_1) = g(x_2) \Rightarrow \begin{cases} 1 = 1 \quad , & x_1 = a = x_2 \\ f(x_1) + 1 = f(x_2) + 1 \ , & x_1 \neq a \ and \ x_2 \neq a \end{cases}$$

$\Longrightarrow f(x_1) + 1 = f(x_2) + 1$

$\Longrightarrow f(x_1) = f(x_2)$

$\Longrightarrow x_1 = x_2$ (f is 1-1)

$\therefore g$ is 1-1

g is onto? $R_g = \{y \in N : \exists x \in A \cup \{a\}, y = g(x)\} = N$

g is onto $\Longrightarrow$ g is bijective

$\therefore A \cup \{a\} \approx N$ $\therefore A \cup \{a\}$ is countably infinite set.

5.28 Remark:

1) Every finite set is countable.

2) Q the set of rational numbers is countably infinite set.

3) R the set of real numbers is uncountable infinite set.

Chapter Six:

Binary Operations العمليات الثنائية

<u>6.1 Definition:</u> Let A be a non-empty set. Any mapping from $A \times A$ into A is called a binary operation on $\boldsymbol{A}$. The binary operation is denoted by the symbols $*$, #, $, o, ….

Mathematically,

$*: A \times A \to A$ is a binary operation iff

1. $*((a,b)) = a * b \in A$ (closure condition)

2. if $a,b,c,d \in A$ s.t. $(a,b) = (c,d)$ then $a * b = c * d$ (well defined condition).

<u>6.2 Example:</u> Let $A = \{0,1,-1\}$, let $*$ be an operation on A such that $a * b = b^2 \quad \forall a,b \in A$. Is $*$ binary operation on A?

Solution:

Closure? Let $a,b \in A \Longrightarrow a * b \in A$?

$\forall a,b \in A \Longrightarrow a * b = b^2 \in A \Longrightarrow *$ is closure.

Well defined? Let $a,b,c,d \in A$ s.t. $(a,b) = (c,d) \Longrightarrow a * b = c * d$?

Since $(a,b) = (c,d) \Longrightarrow a = c \wedge b = d$

$a * b = b^2$ (def. of *)

$= d^2$ $(b = d)$

$= c * d$ (def. of $*$)

$\therefore *$ is well defined

$\therefore *$ is a binary operation on A

<u>6.3 Example:</u> Let $A = \{0,1,-1\}$, let $*$ be an operation on A such that

$a * b = a - b \quad \forall a,b \in A$

Is * binary operation on A?

Solution:

Closure? Let $a, b \in A \Longrightarrow a * b \in A$?

If $a, b \in A \Longrightarrow a * b = a - b \notin A$

Take $a = 1$, $b = -1 \Longrightarrow a * b = a - b = 1 + 1 = 2 \notin A$

$\therefore *$ is not closure

$\therefore$ * is not a binary operation on A

<u>6.4 Example</u>:

Let $A = N$, let # be an operation on N such that $a\#b = a - b \; \forall a, b \in N$

Is # binary operation on N?

Solution: Closure? Let $a, b \in N \Longrightarrow a\#b \in N$?

If $a, b \in N \Longrightarrow a\#b = a - b \notin N$

Take $a = 2,\ b = 5 \Longrightarrow a\#b = a - b = 2 - 5 = -3 \notin N$

$\therefore$ # is not closure

$\therefore$ # is not a binary operation on N

<u>6.5 Example :</u> (H. W.) Let $A = Z$, let $*$ be an operation on Z such that

$a * b = a + b + 1 \; \forall a, b \in Z$ Is $*$ binary operation on Z?

<u>6.6 Example :</u> (H. W.) Let $A = E$, let $*$ be an operation on E such that

$a * b = 2ab \;\; \forall a, b \in E$. Is $*$ binary operation on E?

<u>6.7 Example:</u> (H. W.) Let $A = O$, let $*$ be an operation on O such that

$a * b = a + b \; \forall a, b \in O$. Is $*$ binary operation on O?

<u>6.8 Example:</u>

1. Let $A = N$, $a * b = a + b \; \forall a, b \in N$

   "+" binary operation a on N الجمع عملية ثنائية على مجموعة االعداد الطبيعية

2. "+" is a binary operation on Z, R, Q, E.

3. "-" is a binary operation on Z, R, Q, E.

4. "×" is a binary operation on Z, R,, E, O, N .

5. "÷" is a binary operation on $R\backslash\{0\}$, $Q\backslash\{0\}$.

6. "÷" is not a binary operation on $N\backslash\{0\}$, $Z\backslash\{0\}$.

6.9 Properties of Binary Operations خواص العمليات الثنائية

1.Commutative Binary Operation العملية الثنائية الابدالية

A binary operation $*$ on a set A is called commutative iff $a * b = b * a \, \forall a, b \in A$

6.10 Example:

"+" is a commutative binary operation on N, Z, R,, E

"." is a commutative binary operation on N, Z, R, Q, E

"-" is not commutative binary operation on N, Z, R, Q, E

6.11 Example: Let $a * b = a + b + ab \quad \forall a, b \in Z$. Is $*$ commutative binary operation on Z?

Solution: $*$ binary operation?

Closure? Let $a, b \in Z \Longrightarrow a + b \in Z \Longrightarrow a + b + ab \in Z$

$\therefore$ * is closure

well-defined? Let $a, b, c, d \in Z$ s.t. $(a, b) = (c, d) \implies a * b = c * d$?

Since $(a, b) = (c, d) \Longrightarrow a = c \wedge b = d$

$\Longrightarrow a * b = a + b + ab$ (def. of *)

$= c + d + cd \quad (a = c \wedge b = d)$

$= c * d$

$\therefore *$ is well defined

$\therefore$ * is a binary operation

Commutative? $a * b = a + b + ab = b + a + ba = b * a$

$*$ is commutative.

<u>6.12 Example:</u> Let $a\$b = a \quad \forall a, b \in Q$. Is \$ commutative binary operation on Q?

Solution: \$ binary operation? (H.W.)

Comm.? $a\$b = a$ and $b\$a = b \Longrightarrow a \neq b$

Take $a = \frac{1}{3}$ and $b = 5 \Longrightarrow a\$b = \frac{1}{3}$ and $b\$a = 5$

<u>6.13 Example</u>: Let $A * B = A \cup B \quad \forall A, B \in P(X)$. Is $\cup$ commutative binary operation on $P(X)$?

<u>Solution</u>: $*$ binary operation?

Closure? Let $A, B \in P(X) \Longrightarrow A * B = A \cup B \subseteq X \Longrightarrow A * B \in P(X)$

$\therefore \cup$ is closure.

well-defined? Let $A, B, C, D \in P(X)$ s.t. $(A, B) = (C, D) \Longrightarrow A * B = C * D$?

$(A, B) = (C, D) \Longrightarrow A = C \wedge B = D \Longrightarrow A * B = A \cup B$ (def. of *)

$= C \cup D \quad (A = C \wedge B = D)$

$= C * D$

$\therefore$ * is well defined

$\therefore$ * is a binary operation

Commutative? (H.W.)

<u>2. Associative Binary Operation</u> <u>التجميعية الثنائية العملية</u>

A binary operation $*$ on a set A is called associative if and only if

$$(a * b) * c = a * (b * c) \quad \forall a, b, c \in A$$

<u>6.14 Example:</u> Let $a.b = a + b - 2 \quad \forall a, b \in Z$. Is "." associative, commutative binary operation on ?

Solution: "." binary operation? (H.W.)

Associative? Let $a, b, c \in Z$, $(a.b).c = a.(b.c)$?

$(a.b).c = (a + b - 2).c$ (def. of .)

$= (a + b - 2) + c - 2$ (def. of .)

$= a + b + c - 4 \dots .(1)$

$a.(b.c) = a.(b + c - 2)$ (def. of .)

$= a + (b + c - 2) - 2$ (def. of .)

$= a + b + c - 4 \dots .(2)$

From (1)&(2), $(a.b).c = a.(b.c)$

Commutative? (H.W.)

6.15 Example: (H.W.) Let $A * B = A \cup B \ \forall A, B \in P(X)$. Is $\cup$ associative binary operation on $P(X)$?

6.16 Example: (H.W.) Let $A * B = A \cap B \ \forall A, B \in P(X)$. Is $\cap$ associative, commutative binary operation on $P(X)$?

3. Distributive Property التوزيع خاصية

Let * and # are two binary operations on a set A. Then * is distributive over # from the left if and only if

$$\boldsymbol{a} * (\boldsymbol{b}\#\boldsymbol{c}) = (\boldsymbol{a} * \boldsymbol{b})\#(\boldsymbol{a} * \boldsymbol{c}) \ \forall a, b, c \in A$$

Also, * is distributive over # from the right if and only if

$$(\boldsymbol{b}\#\boldsymbol{c}) * \boldsymbol{a} = (\boldsymbol{b} * \boldsymbol{a})\#(\boldsymbol{c} * \boldsymbol{a}) \ \forall a, b, c \in A$$

6.17 Remark:

1. $a * (b\#c) \neq (b\#c) * a$ (in general)

2. If $a * (b\#c) = (b\#c) * a$ then we say that $*$ **is distributive over** #

6.18 Example: Let $*$ be a binary operation on Z such that

$$a * b = a \ \forall a, b \in Z$$

Let # be a binary operation on Z such that $a\#b = a + b - 2\ \forall a, b \in Z$

Is * distributive over # from left and from right?

*** distributive over # from left ?** We must show

if $a * (b\#c) = (a * b)\#(a * c)\ \forall a, b, c \in Z$

$a * (b\#c) = a\ (def. of\ *)$…..(1)

$(a * b)\#(a * c) = a\#a$ (def. of *)

$= a + a - 2$ (def. of #)

$= 2a - 2$ ….(2)

From (1) and (2), $a * (b\#c) \neq (a * b)\#(a * c)$

∴ * is not distributive over # from left

*** distributive over # from right ?** We must show

if $(b\#c) * a = (b * a)\#(c * a)\ \forall a, b, c \in A$

$(b\#c) * a = b\#c$ (def. of *)

$= b + c - 2$ (def. of #) ….(1)

$(b * a)\#(c * a) = b\#c$ (def. of *)

$= b + c - 2$ (def. of #) ….(2)

From (1) and (2), $(b\#c) * a = (b * a)\#(c * a)$

∴ * is distributive over # from right

<u>6.19 Example:</u> (H.W.) Let $*$ be a binary operation on N such that $a * b = ab\ \forall a, b \in N$

Let # be a binary operation on N such that $a\#b = a + b\ \forall a, b \in N$

Is * distributive over # from left and from right?

6.20 Example: (H.W.) Let $*$ be a binary operation on (X) such that

$$A * B = A \cup B \ \forall A, B \in P(X)$$

Let # be a binary operation on (X) such that $A\#B = A \cap B \quad \forall A, B \in P(X)$ Is * distributive over # from left and from right?

6.21 Definition: The Identity Element العنصر المحايد

Let $*$ be a binary operation on a set A and $e \in A$, then e is called the identity element of $\boldsymbol{A}$ if and only if $a * e = e * a = a \ \forall a \in A$

6.22 Example:

1."0" is the identity element of the sets Z, Q, R with respect to (w.r.t.) (+)

الصفر هو العنصر المحايد للمجموعات Z, Q, R بالنسبة لعملية الجمع

$i.e.$, $\mathrm{a} + 0 = 0 + \mathrm{a} \ \forall \mathrm{a} \in \mathrm{Z}, \mathrm{Q}, \mathrm{R}$

2."0" is not the identity element of the sets Z,, R with respect to (w.r.t.) (-)

الصفر لايمثل العنصر المحايد للمجموعات Z,Q, R بالنسبة لعملية الطرح

$i.e.$, $\exists \mathrm{a} \in \mathrm{N}, \mathrm{Z}, \mathrm{Q}, \mathrm{R} \ \mathrm{s.t.} \ \mathrm{a} - 0 \neq 0 - \mathrm{a}$

3. "1" is the identity element of the sets N, Z, Q, R w.r.t. (.)

الواحد هو العنصر المحايد للمجموعات Z,Q, R بالنسبة لعملية الضرب

$i.e.$, $a.1 = 1.a \ \forall a \in N, Z, Q, R$

4. "1" is not the identity element of the sets $Q - \{0\}, R - \{0\}$ with respect to (w.r.t.) (/)

الواحد لايمثل العنصر المحايد للمجموعات $Q - \{0\}$, $R - \{0)$, بالنسبة لعملية القسمة

$i.e.$, $\exists a \in Q - \{0\}, R - \{0\} \ s.t. \frac{a}{1} \neq \frac{1}{a}$

6.23 Example: Let # be a binary operation on $R\backslash\{-1\}$ such that

$a\#b = a + b + ab \ \forall a, b \in R\backslash\{-1\}$.

Find the identity element of $R\backslash\{-1\}$ with respect to #.

Solution: Let e be the identity element of E s.t.

$a\#e = e\#a = a \; \forall a \in R\backslash\{-1\}$

We must find e?

$a\#e = a \Rightarrow a + e + ae = a$ (def. of #)

$\Rightarrow e + ae = 0 \Rightarrow e(1 + a) = 0$

Either $e = 0 \; or \; 1 + a = 0$

$\Rightarrow a = -1 \notin R\backslash\{-1\}$ يهمل

$\therefore e = 0 \in R\backslash\{-1\}$ ….(1)

$e\#a = a \Rightarrow e + a + ea = a$ (def. of #)

$\Rightarrow e + ea = 0 \Rightarrow e(1 + a) = 0$

Either $e = 0 \; or \; 1 + a = 0$

$\Rightarrow a = -1 \notin R\backslash\{-1\}$ يهمل

$\therefore e = 0 \in R\backslash\{-1\}$….(2)

From (1) and (2), $e = 0$

6.24 Example: (H. W.) Let $*$ be a binary operation on N such that

$a * b = a + b + ab \; \forall a, b \in N.$

Find the identity element of N with respect to *.

6.25 Example : (H. W.) Let $*$ be a binary operation on N such that

$a * b = a + b - 1 \; \forall a, b \in N.$

Find the identity element of N with respect to *.

6.26 Example: Let $*$ be a binary operation on P(X) such that

$A * B = A \cup B \; \forall A, B \in P(X).$

Find the identity element of (X) with respect to *.

Solution: Let e be the identity element of (X) s.t.

$A * e = e * A = A \quad \forall A \in P(X)$

$e = \emptyset$ because $A \cup \emptyset = \emptyset \cup A = A \quad \forall A \in P(X)$

6.27 Example: (H. W.) Let $*$ be a binary operation on (X) such that

$A * B = A \cap B \; \forall A, B \in P(X)$. Find the identity element of (X) with respect to *.

6.28 Theorem: Let e is the identity element of a set A with respect to *, then e is unique.

Proof: Let e is the identity element of a set A with respect to *

Suppose e' is another identity of A w.r.t. *

Since e is the identity $\Longrightarrow e * e' = e' * e = e'$….(1)

Since e' is the identity $\Longrightarrow e' * e = e * e' = e$….(2)

From (1) and (2), $e = e'$

$\therefore e$ is unique

6.29 Definition: The Inverse Element العنصر النظير

Let $*$ be a binary operation on a set A and e is the identity element of A. Let $a \in A$, then $b \in A$ is called the inverse element of $\boldsymbol{a}$ if and only if $a * b = b * a = e$.

The inverse element b is denoted by a^{-1} . So

$$a * a^{-1} = a^{-1} * a = e$$

6.30 Example: Find the inverse element of each element in Z,, R w.r.t "+"

Solution: The identity element $e = 0$

$a * a^{-1} = 0 \Longrightarrow a + a^{-1} = 0 \Longrightarrow a^{-1} = -a \quad \forall a \in Z, Q, R$

AND, $a^{-1} * a = 0 \Longrightarrow a^{-1} + a = 0 \Longrightarrow a^{-1} = -a \quad \forall a \in Z, Q, R$

$\therefore a^{-1} = -a \quad \forall a \in Z, Q, R$

6.31 Example: Find the inverse element of each element in $Q\backslash\{0\}, R\backslash\{0\}$ w.r.t "."

Solution: The identity element $e = 1$

$a * a^{-1} = 0 \Rightarrow a.a^{-1} = 1 \Rightarrow a^{-1} = \frac{1}{a} \ \forall a \in Q\backslash\{0\}, R\backslash\{0\}$

And , $a^{-1} * a = 0 \Rightarrow a^{-1}.a = 1 \Rightarrow a^{-1} = \frac{1}{a} \ \forall a \in Q\backslash\{0\}, R\backslash\{0\}$

$\therefore a^{-1} = \frac{1}{a} \ \forall a \in Q\backslash\{0\}, R\backslash\{0\}$

6.32 Example: Let # be a binary operation on $Z\backslash\{-1\}$ such that

$a\#b = a + b + ab \ \forall a, b \in Z\backslash\{-1\}$. Find the inverse element of each element in $Z\backslash\{-1\}$ (if exist).

Solution: From Example, $e = 0$

Let $a \in Z\backslash\{-1\}$ and a^{-1} is the inverse of a

$\Rightarrow a * a^{-1} = a^{-1} * a = e \ a * a^{-1} = e$

$\Rightarrow a + a^{-1} + aa^{-1} = 0 \Rightarrow a + a^{-1}(1 + a) = 0$

$\Rightarrow a^{-1} = -\frac{a}{1+a}$

$a^{-1} * a = e \Rightarrow a^{-1} + a + a^{-1}a = 0$

$\Rightarrow a + a^{-1}(1 + a) = 0 \Rightarrow$

$$a^{-1} = -\frac{a}{1 + a} \in Q$$

بصورة عامة نظير كل عدد في $Z\backslash\{-1\}$ هو عدد نسبي ماعدا -2 ، $a = 0$ فان نظيرهما عدد صحيح

If $a = 0 \Rightarrow a^{-1} = 0 \in Z\backslash\{-1\} \Rightarrow 0^{-1} = 0$

If $a = -2 \Rightarrow a^{-1} = \frac{2}{-1} = -2 \Rightarrow a^{-1} = -2 \in Z\backslash\{-1\}$

If $a = 3 \Rightarrow a^{-1} = -\frac{3}{4} \notin Z\backslash\{-1\}$

$\therefore a = 3$ has no inverse

$\therefore \forall a \neq 0, -2, \quad a^{-1} \notin Z\backslash\{-1\}$

6.33 Example: (H. W.) Let $*$ be a binary operation on $Q\{0\}$ such that $a * b = 2ab \ \forall a, b \in Q\backslash\{0\}$. Find the identity element of $Q\backslash\{0\}$ with respect to *.

Find the inverse of each element in $Q\backslash\{0\}$ (if exist).

Example: (H. W.) Let $*$ be a binary operation on Z such that $a * b = a + b + 5 \quad \forall a, b \in Z$. Find the inverse of each element of Z with respect to *.

6.34 Group الزمرة

Let G be a non empty set and * be a binary operation on G. The pair $(G,*)$ is called **group** if and only if * is associative, there is an identity element and each element has an inverse.

Mathematically,

(G,*) is called group iff

1. $G \neq \emptyset$

2. $*$ is a binary operation on G

3. $*$ is associative on G

4. $\exists$ identity element $e \in G$ s.t. $a * e = e * a = a$

5. $\forall a \in G, \exists \ a^{-1} \in G$ s.t. $a * a^{-1} = a^{-1} * a = e$

6.35 Remark: If $(G,*)$ is a group and * is a commutative then $(G,*)$ is called **commutative group**.

Mathematically,

A group $(G,*)$ is called commutative iff $a * b = b * a \ \forall a, b \in G$

6.36 Example: Show that $(Z, +)$ is a commutative group

1. $Z \neq \emptyset$

2. + is associative binary operation on Z

3. $\exists e = 0 \in Z \ s.t. \ a + 0 = 0 + a = a \ \forall a \in Z$

4. $\exists \quad a^{-1} = -a \in Z \, \forall a \in Z \, s.t. \, a + a^{-1} = a^{-1} + a = 0$

$\therefore (Z, +)$ is a group

$\forall a, b \in Z, \quad a + b = b + a$

$\therefore (Z, +)$ is a commutative group

6.37 Example:

$(Q, +)$ is a comm. group

$(R, +)$ is a comm. group

$(N, +)$ is not a group

$(Z, .)$ is not a group

$(O, +)$ is not a group

$(R\backslash\{0\}, .)$ is a group

$(R, .)$ is not a group

6.38 Example:

Show that $(Z,*)$ is a commutative group such that $a * b = a + b - 5$

Solution:

1.Closure: let $a, b \in Z \Longrightarrow a * b = a + b - 5 \in Z$

$\Longrightarrow$ closure is true

well-defined: let $a, b, c, d \in Z$ s.t. $(a, b) = (c, d) \Longrightarrow a * b = c * d$?

Since $(a, b) = (c, d) \Longrightarrow a = c \wedge b = d$

$\Longrightarrow a * b = a + b - 5$ (def. of *)

$= c + d - 5$ ($a = c \wedge b = d$)

$= c * d$

$\therefore *$ is well defined

$\therefore$ * is a binary operation

2. **associative** (H.W.)

3. **Identity:** let $a \in Z$ we find $e \in Z$ such that $a * e = e * a = a$

$a * e = a$

$\Longrightarrow a + e - 5 = a$

$\Longrightarrow e = 5 \in Z$ …(1)

Similarly, $e * a = a$

$\Longrightarrow e + a - 5 = a$

$\Longrightarrow e = 5 \in Z$ …(2)

From (1) &(2) , $e = 5$

4. Inverse: $\forall a \in Z$, we find $a^{-1} \in Z$ such that $a * a^{-1} = a^{-1} * a = e$

$a * a^{-1} = e$

$\Longrightarrow a + a^{-1} - 5 = 5$

$\Longrightarrow a^{-1} = 10 - a \in Z$ …(1)

Similarly, $a^{-1} * a = e$

$\Longrightarrow a^{-1} + a - 5 = 5$

$\Longrightarrow a^{-1} = 10 - a \in Z$ …(2)

From (1) &(2) , $a^{-1} = 10 - a$

$\therefore (Z,*)$ is a group

Commutative: (H.W.)

<u>6.39 Example:</u> Is $(P(X),\cup)$ group?

<u>Solution</u>:

1. $\cup$ is a binary operation (مثال سابق)

2. $\cup$ is associative (مثال سابق)

3. $\exists \emptyset \in P(X)$ s.t. $A \cup \emptyset = \emptyset \cup A = A$

$\therefore e = \emptyset$

4. **Inverse**: $\forall A \in P(X)$, we find $A^{-1} \in P(X)$ such that

$A \cup A^{-1} = A^{-1} \cup A = \emptyset$

If $A = \emptyset$ then $A^{-1} = \emptyset$ s.t. $A \cup A^{-1} = \emptyset$

When $A \neq \emptyset$ then there is no inverse to A

المجموعة الوحيدة التي يوجد لها نظير هي ال $\emptyset$

$\therefore (P(X),\cup)$ is not a group

6.40Example: (H.W.) Is $(P(X),\cap)$ group?

Is $(P(X),\backslash)$ group?

6.41 Example: Let $F(A) = \{f, f: A \to A \text{ is bijective map.}\}$

let $*$ be an operation on $F(A)$ s.t. $f * g = fog$

Is $(F(A),*)$ commutative group?

Solution:

1.Closure: let $f, g \in F(A) \Rightarrow f * g = fog \in F(A)$?

$f \in F(A) \Rightarrow f: A \to A$ is bijective

$g \in F(A) \Rightarrow g: A \to A$ is bijective

$\therefore fog: A \to A$ is bijective

Closure is true

well-defined: let $f_1, f_2, g_1, g_2 \in F(A)$ s.t. $(f_1, f_2) = (g_1, g_2) \Rightarrow$

$f_1 * f_2 = g_1 * g_2$?

Since $(f_1, f_2) = (g_1, g_2) \Rightarrow f_1 = g_1 \wedge f_2 = g_2$

$\Rightarrow f_1 * f_2 = f_1 o f_2$ (def. of *)

$= g_1 o g_2$ ($f_1 = g_1 \wedge f_2 = g_2$)

$= g_1 * g_2$

$\therefore *$ is well defined

$\therefore$ * is a binary operation

2. associative: $\forall f, g, h \in F(A)$

$(fog)oh = fo(goh)$ (by theorem, chapter4)

3. Identity: $\exists i_A: A \rightarrow A$ is bijective such that $foi_A = i_A of = f$

$\forall f \in F(A)$(by theorem, ch4)

$\therefore e = i_A$

4. Inverse: $\forall f \in F(A) \Longrightarrow f: A \rightarrow A$ is bijective

$\exists f^{-1} : A \rightarrow A$ is bijective $\Longrightarrow f^{-1} \in F(A)$

Such that $fof^{-1} = f^{-1}of = i_A$ (by theorem, ch4)

$\therefore$ ((A), o) is a group Commutative: Since f: $A \rightarrow A$ and g: $A \rightarrow A$ fog = gof
$\therefore (F(A), o)$ is a commutative group

6.42 Semi Group شبه الزمرة

Let A be a non-empty set and * be a binary operation on A. The pair $(A,*)$ is called semi group if and only if * is associative.

Mathematically, $(A,*)$ is called semi group iff

1. $A \neq \emptyset$

2. $*$ is a binary operation on A

3. $*$ is associative on A

6.43 Example:

$(N, +)$ is a semi group but not a group

(Z, .) is a semi group but not a group

6.44 Remark: Every group is a semi group

6.45 Ring الحلقة

Let R be a non empty set and * and # be two binary operations on R. The ordered triple $(R,*,\#)$ is called ring if and only if

1. $R \neq \emptyset$

2. $(R,*)$ is a commutative group

3. $(R,\#)$ is a semi group

4. # is distributed over $*$ (from left and right)

6.46 Example: $(Z,+,.)$ is a ring

1. $(Z,+)$ is a commutative group

2. $(Z,.)$ is a semi-group

3. $a.(b + c) = a.b + a.c \;\forall a,b,c \in R$ (distribution from left)

$(b + c).a = b.a + c.a \;\forall a,b,c \in R$ (distribution from right)

6.47 Example: $(Q,+,.)$ is a ring

$(R,+,.)$ is a ring

6.48 Commutative Ring: الحلقة الابدالية

A ring $(R,*,\#)$ is called commutative iff $a\#b = b\#a \quad \forall a,b \in R$

الابدال يجب ان يتحقق على العملية الثانية

6.49 Example: $(Z,+,.)$ is a commutative ring because $a.b = b.a \;\forall a, \in Z$

$(Q,+,.)$ is a commutative ring

$(R,+,.)$ is a commutative ring

6.50 Ring with Identity Element الحلقة ذات العنصر المحايد

A triple $(R,*,\#)$ has an identity element with respect to (#) if and only if

$$a\#e = e\#a = a \;\forall a \in R$$

6.51 Example:

$(Z,+,.),(Q,+,.),(R,+,.)$ are rings with $e=1$ because $a.1=1.a$

6.52 Ordered Ring **الحلقة المرتبة**

A triple $(R,*,\#)$ is called **totally ordered ring** if and only if there is a totally ordered relation such that

1. $(R,*,\#)$ is a ring

2. The relation R is totally ordered relation $T.O.R$

3. $\forall a,b \in R\ if\ a\ b\ then\ a*c\ b*c, \forall c \in Z$

4. $\forall a,b \in R\ if\ a\ b\ then\ a\#c\ b\#c, \forall c \geq 0$

The totally ordered relation is denoted by $(R,*,\#,)$

6.53 Example: $(Z,+,.,\leq)$ is a totally ordered ring since

1. $(Z,+,.)$ is a ring (مثال سابق)

2. $(\mathrm{Z},\leq)$ is $\mathrm{T.O.R}$ (see Example, Chapter 3)

3. $\forall \mathrm{a,b} \in \mathrm{Z}$ and $\mathrm{c} \in \mathrm{Z}$ if $\mathrm{a} \leq \mathrm{b}$ T.P. $\mathrm{a+c} \leq \mathrm{b+c}$

Let $\mathrm{a} \leq \mathrm{b} \Rightarrow \mathrm{a}=\mathrm{b}-\mathrm{r}, \quad \mathrm{r} \geq 0$

$\Rightarrow \mathrm{a+c}=(\mathrm{b+c})-\mathrm{r}, \quad \mathrm{c} \in \mathrm{Z}$ and $\mathrm{r} \geq 0$

$\Rightarrow \mathrm{a+c} \leq \mathrm{b+c}$

4. $\forall \mathrm{a,b} \in \mathrm{Z}$ and $\mathrm{c} \geq 0$, if $\mathrm{a} \leq \mathrm{b}$ then $\mathrm{a.c} \leq \mathrm{b.c}$

Let $\mathrm{a} \leq \mathrm{b} \Rightarrow \mathrm{a}=\mathrm{b}-\mathrm{r}, \mathrm{r} \geq 0$
$\Rightarrow \mathrm{a.c}=\mathrm{b.c}-\mathrm{cr}, \quad \forall \mathrm{c} \geq 0$
$\Rightarrow \mathrm{a.c}=\mathrm{b.c}-\mathrm{cr}, \quad \mathrm{cr} \geq 0$
$\Rightarrow \mathrm{a.c} \leq \mathrm{b.c}$

6.54 Example: (H.W.) Show that $(Z,+,.,\geq)$ is a totally ordered ring