Series

Def. Let $\langle a_n \rangle$ be a sequence in $\mathbb{R}$, the following sums $a_1 + a_2 + a_3 + \cdots$ is called an infinite series and is denoted by $\sum a_n$ or $\sum_{n=1}^{\infty} a_n$.

For a given Series $\sum_{n=1}^{\infty} a_n$ we define the sequence of partial sums $\langle S_n \rangle$ by

$$S_1 = a_1$$
$$S_2 = a_1 + a_2$$
$$S_n = a_1 + a_2 + \cdots + a_n = \sum_{i=1}^{n} a_i$$

Def. An infinite series $\sum_{n=1}^{\infty} a_n$ is said to be convergent if the seq. $\langle S_n \rangle$ of partial sums is convergent

Notes:

1) If $\langle S_n \rangle$ converge to the real Number l then the Sum of the Series $\sum_{n=1}^{\infty} a_n$ is l i.e $\sum_{n=1}^{\infty} a_n = l$.

2) The Series $\sum_{n=1}^{\infty} a_n$ is said to be divergent (to ∞ or $-\infty$) or oscillatory according as seq. $\langle S_n \rangle$ of its partial sums divergent to (∞ or $-\infty$) or oscillatory.

Theorem: Necessary condition for the convergence of a series

If $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$

proof: let $\langle S_n \rangle$ be the seq. of partial sums of the series $\sum a_n$. Then $\langle S_n \rangle$ is convergent.

Let $\lim_{n\to\infty} S_n = \ell$. Since $a_n = S_n - S_{n-1}$, $\forall n$

We have $\lim_{n\to\infty} a_n = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = \ell - \ell = 0$.

Q/ Is the converse of the above theorem true? H.W

EX. The series $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots + \frac{n}{n+1} + \cdots$

is not convergent since $\lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0$.

Q/ Test the convergence of the following series

1) $\sum_{n=1}^{\infty} (n)^{\frac{1}{n}}$ 2) $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^n$ 3) $3-3+3-3+\cdots$

Note: The series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent (Harmonic Series)

Def: A series $\sum_{n=1}^{\infty} a_n$ is said to be positive term series

if $a_n \geq 0$, $n \in \mathbb{N}$

Note:

بما أن الحد $a_n = 0$ لا يؤثر على تقارب وتباعد المتسلسلات.

لذلك بدون فقدان عمومية التعريف، المتسلسلة ذات الحدود الموجبة

هي التي يكون فيها $a_n > 0$ لكل n.

EX. The Series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is conv.

$$a_1 = \frac{1}{1(1+1)} = \frac{1}{1(2)} = 1 - \frac{1}{2}$$

$$a_2 = \frac{1}{2(2+1)} = \frac{1}{2(3)} = \frac{1}{2} - \frac{1}{3}$$

$$a_3 = \frac{1}{3(3+1)} = \frac{1}{3(4)} = \frac{1}{3} - \frac{1}{4}$$

$$a_4 = \frac{1}{4(4+1)} = \frac{1}{4(5)} = \frac{1}{4} - \frac{1}{5}$$

$$\vdots$$

$$a_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$S_1 = a_1 = 1 - \frac{1}{2}$$

$$S_2 = a_1 + a_2 = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) = 1 - \frac{1}{3}$$

$$S_3 = a_1 + a_2 + a_3 = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) = 1 - \frac{1}{4}$$

$$S_4 = a_1 + a_2 + a_3 + a_4 = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) = 1 - \frac{1}{5}$$

$$\vdots$$

$$S_n = a_1 + a_2 + a_3 + \cdots + a_n = 1 - \frac{1}{n+1}$$

The seq. of partial sums of $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is

$\langle s_n \rangle = \langle 1 - \frac{1}{n+1} \rangle$, $\langle s_n \rangle$ is conv. since

$\lim_{n\to\infty} (1 - \frac{1}{n+1}) = 1 \Rightarrow \sum a_n$ is conv.

Harmonic Series

$\sum \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$ is divergent series since

$s_1 = a_1 = 1$

$s_2 = a_1 + a_2 = 1 + \frac{1}{2}$

$s_3 = a_1 + a_2 + a_3 = 1 + \frac{1}{2} + \frac{1}{3}$

$\vdots$

$s_n = a_1 + a_2 + \cdots + a_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$

$s_{n+1} = a_1 + a_2 + \cdots + a_n + a_{n+1} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \frac{1}{n+1}$

$s_{n+2} = a_1 + a_2 + \cdots + a_n + a_{n+1} + a_{n+2} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2}$

$\vdots$

$s_{n+n} = s_{2n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}$

let $m = 2n$

$$|s_m - s_n| = \left|(1 + \tfrac{1}{2} + \cdots + \tfrac{1}{n} + \tfrac{1}{n+1} + \tfrac{1}{n+2} + \cdots + \tfrac{1}{2n}) - (1 + \tfrac{1}{2} \cdots + \tfrac{1}{n})\right|$$

$$= \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}$$

$$> \frac{1}{2n} + \frac{1}{2n} + \cdots + \frac{1}{2n}$$

$$= n \cdot \frac{1}{2n} = \frac{1}{2}$$

If $\varepsilon = \frac{1}{2} \Rightarrow |S_m - S_n| > \varepsilon$

then $\langle S_n \rangle$ is not a cauchy seq.

$\Rightarrow \langle S_n \rangle$ is not convergent seq.

$\Rightarrow \sum_{n=1}^{\infty} a_n$ is divergent

Geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots$$

where $a > 0$, r is called the base of series

The sequence of partial sums is

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1}$$

1. if $|r| = 1$

$\Rightarrow S_n = a + a + a + \cdots + a = na$

$\langle S_n \rangle = \langle na \rangle$ divergent $\Rightarrow \sum_{n=1}^{\infty} ar^{n-1}$ divergent

2. if $|r| < 1$

$\Rightarrow S_n = a + ar + ar^2 + \cdots + ar^{n-1}$

$\Rightarrow rS_n = ar + ar^2 + ar^3 + \cdots + ar^n$

$\Rightarrow S_n - rS_n = a - ar^n$

$\Rightarrow S_n(1-r) = a(1-r^n)$

$\Rightarrow S_n = \dfrac{a(1-r^n)}{1-r}$

when $n \to \infty \Rightarrow \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r}$

$$= \frac{a(1-0)}{1-r} = \frac{a}{1-r}$$

Thus $\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$ conv.

3. If $|r|>1$

$\Rightarrow S_n = \frac{a(1-r^n)}{1-r}$

when $n \to \infty \Rightarrow r^n = \pm\infty \Rightarrow S_n \to \infty$

Thus $\sum_{n=1}^{\infty} ar^{n-1}$ divergent

$$\sum_{n=1}^{\infty} ar^{n-1} = \begin{cases} \text{divergent} & \text{if } |r| \geq 1 \\ \text{convergent}, \ \sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} & \text{if } |r|<1 \end{cases}$$

EX.

1) $\sum a_n = 1 + \frac{7}{3} + \left(\frac{7}{3}\right)^2 + \left(\frac{7}{3}\right)^3 + \cdots$ Geometric Series

$a=1$, $r=\frac{7}{3}$, $|r| = \left|\frac{7}{3}\right| = \frac{7}{3} > 1$, then $\sum a_n$ div.

2) $\sum a_n = 1 - \frac{2}{3} + \frac{4}{9} - \frac{8}{27} + \frac{16}{81} - \cdots$ H.W

Theorem: If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent series and $k \in \mathbb{R}$, then:

1) $\sum_{n=1}^{\infty} (a_n + b_n)$ convergent and $\sum_{n=1}^{\infty}(a_n+b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

2) $\sum_{n=1}^{\infty} k a_n$ convergent and $\sum_{n=1}^{\infty} k a_n = k \sum_{n=1}^{\infty} a_n$

proof: 1) let $\langle s_n \rangle$ be a seq. of partial sums of $\sum a_n$ and $\langle t_n \rangle$ be a seq. of partial sums of $\sum b_n$.

Since $\sum a_n$ conv., then $\exists\, s \in \mathbb{R} \ni \sum a_n = s$ and $s_n \to s$

Since $\sum b_n$ conv., then $\exists\, t \in \mathbb{R} \ni \sum b_n = t$ and $t_n \to t$

$\lim_{n\to\infty} (s_n + t_n) \longrightarrow s+t$, but $\langle s_n + t_n \rangle$ is the seq. of partial sums of $\sum_{n=1}^{\infty}(a_n+b_n) \Rightarrow \sum_{n=1}^{\infty}(a_n+b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n = s+t$

2) H.W

Exc. Give an example for two divergent series but their sum is convergent series.

Theorem: A positive term series $\sum a_n$ is conv. iff its seq. of partial sum $\langle s_n \rangle$ is bdd above.

proof: H.W

Series Test

1- Comparison test

Theorem: let $\sum a_n$ and $\sum b_n$ be two positive term series such that $a_n \leq b_n$ then

a) $\sum b_n$ conv. $\Rightarrow \sum a_n$ conv.

b) $\sum a_n$ div. $\Rightarrow \sum b_n$ div.

proof: a) let $\sum b_n$ conv. T.p $\sum a_n$ conv., let $S_n = a_1 + \dots + a_n$, since $a_n \leq b_n \ \forall n$ i.e S_n is bdd above, $\sum b_n = S$, i.e S is an upper bound to S_n. Since $a_n \geq 0 \ \forall n \Rightarrow S_n$ is increasing seq. $\Rightarrow S_n$ conv. i.e $\sum a_n$ conv.

b) We have $\sum a_n$ div. series T.p $\sum b_n$ div. if $\sum b_n$ conv. and $b_n \geq a_n \geq 0 \Rightarrow \sum a_n$ conv. and this contradicts with assumption that $\sum a_n$ div.

2- limit comparison test.

Theorem: let $\sum a_n$ and $\sum b_n$ be a positive term series

1) If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ (finite and positive) then either both series conv. or both series div.

2) If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ is conv. then $\sum a_n$ conv.

3) If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ div. then $\sum a_n$ is div.

3- p- Series

Theorem: The series $\sum \frac{1}{n^p}$ converges for $p>1$ and diverges for $p \leq 1$.

if $p=1$ then $\sum \frac{1}{n}$ (harmonic series) is div.

if $p<1$ then $n^p \leq n \Rightarrow \frac{1}{n^p} \geq \frac{1}{n} \Rightarrow \sum \frac{1}{n^p}$ div.

if $p>1$ H.W

EX. Test the convergence of the series $\sum_{n} (\sqrt{n+1} - \sqrt{n})$

Sol. write $a_n = \sqrt{n+1} - \sqrt{n} = \frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+1} + \sqrt{n}}$

$$= \frac{1}{\sqrt{n+1} + \sqrt{n}}$$

$$= \frac{1}{\sqrt{n}} \left(\frac{1}{\sqrt{1+\frac{1}{n}} + 1} \right)$$

Take $b_n = \frac{1}{\sqrt{n}}$, $n \in \mathbb{N}$

Then $\lim_{n \to \infty} \frac{a_n}{b_n} = \frac{1}{2} \neq 0$. Then by limit comparison test, $\sum a_n$ and $\sum b_n$ conv. or div. together. But $\sum b_n = \sum \frac{1}{\sqrt{n}}$ is div. by p-Series test, hence

$\sum a_n = \sum (\sqrt{n+1} - \sqrt{n})$ is div.

Alternating Series

Def. A series of the form $\sum_n (-1)^{n-1} a_n = a_1 - a_2 + a_3 - \cdots \ (a_n > 0)$

is called an alternating series

In the other words, a series whose terms are alternatively positive and negative is called an alternative series

one may observe that the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

is an alternative, whereas $1 + \frac{1}{2} + \frac{1}{3} - \frac{1}{4} - \frac{1}{5} - \frac{1}{6} + \frac{1}{7} + \cdots$

is not an alternating Series.

Leibnitz Test.

let $\sum (-1)^{n-1} a_n$ be an alternating series such that

a) $a_{n+1} \leqslant a_n \quad \forall\, n \in \mathbb{N}$

b) $\lim_{n \to \infty} a_n = 0.$

Then the series is convergent

EX. the alternating series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ is conv.

Absolute and Conditional Convergence

Def. A series $\sum a_n$ is said to be absolutely convergent if the series $\sum |a_n|$ is convergent

EX. the series $\sum (-1)^{n+1} \frac{1}{n^3} = 1 - \frac{1}{2^3} + \frac{1}{3^3} - \frac{1}{4^3} + \cdots$

is absolutely convergent since the series $\sum_n |(-1)^n \frac{1}{n^3}| = \sum_n \frac{1}{n^3}$

is conv. by p-test

H.W Give an example of a conv. alternating series but Not absolutely convergent

Theorem: An absolutely convergent series is always convergent.

The converse need not be true.

Def. An infinite series $\sum a_n$ is said to be conditionally convergent if it is convergent but not absolutely convergent.

EX. The alternating series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ is conditionally convergent.

Def. $$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

The series e is conv. since

$$S_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}$$

$$= 1 + 1 + \frac{1}{2\times 1} + \frac{1}{3\times 2\times 1} + \cdots + \frac{1}{n!}$$

$$= 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \cdots + \frac{1}{n!}$$

$$< 1 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^{n-1}}$$

$$= 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{n-1}}$$

متسلسلة هندسية $r = \frac{1}{2}$

$a = \frac{1}{2}$ ومجموعها 1

Then $S_n < 1 + 1 + 1 = 3 \Rightarrow \langle S_n \rangle$ is bdd and increasing

$\Rightarrow \langle S_n \rangle$ conv. $\Rightarrow \sum \frac{1}{n!}$ conv.

H.W: prove that $e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$.

product of series

If $\sum a_n$, $\sum b_n$ conv. series, we see that

$$\sum a_n \cdot \sum b_n = (a_1 + a_2 + \cdots)(b_1 + b_2 + \cdots)$$
$$= a_1(b_1 + b_2 + \cdots) + a_2(b_1 + b_2 + \cdots) + \cdots$$

Def. (Cauchy product of series)

Let $\sum_{n=0}^{\infty} a_n$, $\sum_{n=0}^{\infty} b_n$ be two series and

$c_n = \sum_{k=0}^{n} a_k \cdot b_{n-k} = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0$. We say that

$\sum_{n=0}^{\infty} c_n$ is the product of $\sum a_n$ and $\sum b_n$

$n = 0 \Rightarrow c_0 = \sum_{k=0}^{0} a_k \cdot b_{n-k} = a_0 b_0$, $n = 1 \Rightarrow c_1 = \sum_{k=0}^{1} a_k \cdot b_{n-k} = a_0 b_1 + a_1 b_0$

$n = 2 \Rightarrow c_2 = \sum_{k=0}^{2} a_k \cdot b_{n-k} = a_0 b_2 + a_1 b_1 + a_2 b_0 \quad \ldots$

$$\sum_{n=0}^{\infty} a_n \sum_{n=0}^{\infty} b_n = c_0 + c_1 + \cdots = a_0 b_0 + (a_0 b_1 + a_1 b_0) + (a_0 b_2 + a_1 b_1 + a_2 b_0) + \cdots$$

Theorem: cauchy criterion of convergence of a series

A series $\sum a_n$ is convergent iff for every $\varepsilon > 0$ there exists a positive integer m $\ni$

$$|a_{m+1} + a_{m+2} + \cdots + a_n| < \varepsilon \quad \forall\, n \geq m$$

proof: let $\langle S_n \rangle$ be the seq. of partial sum of the series $\sum a_n$ then $\sum a_n$ is conv.

$\Leftrightarrow \langle S_n \rangle$ is conv. $\Leftrightarrow \langle S_n \rangle$ is a cauchy seq

$\Leftrightarrow \forall\, \varepsilon > 0\ \exists\, m \in \mathbb{Z}^+ \ni |S_n - S_m| < \varepsilon \quad \forall\, n \geq m$

i.e $|a_{m+1} + a_{m+2} + \cdots + a_n| < \varepsilon \quad \forall\, n \geq m$.

sequence and series of functions

Def. A seq. of function $\{f_n\}_{n\in\mathbb{N}}$ is a list of functions $(f_1, f_2, \ldots)$ such that each f_n maps a given subset D of $\mathbb{R}$ into $\mathbb{R}$. $f_n : D \subseteq \mathbb{R} \to \mathbb{R}$

Def. point wise convergence

let $D \subseteq \mathbb{R}$ and $\{f_n\}$ be a seq. of func. defined on D we say that $\{f_n\}$ converges point wise on D to a function $f: D \to \mathbb{R}$ if

$\lim_{n\to\infty} f_n(x)$ exists $\forall\, x \in D$ i.e

$\forall\, \varepsilon > 0\ \exists\, k \in \mathbb{Z}^+ \ni |f_n(x) - f(x)| < \varepsilon \quad \forall\, n > k$.

$f(x)$ is called the point wise limit of the seq. $\{f_n\}$

EX. let $\{f_n\}$ be the seq. of func. on $\mathbb{R}$ defined by $f_n(x)=nx$

This seq. does not conv. point wise on $\mathbb{R}$ because

$\lim_{n\to\infty} f_n(x) = \infty$ for any $x>0$.

EX. let $\{f_n\}$ be a seq. of func. on $\mathbb{R}$ defined by $f_n(x) = \frac{x}{n}$

This seq. is conv. point wise to the zero on $\mathbb{R}$.

Def. let $\{f_n\}$ be a seq. of func. and let $S_n(x) = \sum_{i=1}^{n} f_i(x)$ be the n^{th} partial sum of the infinite series $\sum_{n=1}^{\infty} f_n(x)$

we say $\sum f_n$ converges point wise to f on D if $S_n \to f$ point wise on D. in this case we write $\sum f_n = f$ point wise on D.

EX. let $f_n(x) = x^n$, $-1<x<1$ and let $f(x) = \frac{x}{1-x}$

then $\sum_{n=1}^{\infty} f_n = f$ point wise on $(-1,1)$.