Proof: $E\ (x^k) = \int_0^1 x^k\ f(x)dx = \frac{\Gamma_{\propto +B}}{\Gamma_\propto \Gamma_B}\ \int_0^1 x^{\propto 1-\ k-1}\ (1-x)^{\beta-1}dx$

$= \frac{\Gamma_{\propto +\beta}}{\Gamma_\propto \Gamma_\beta}\ \frac{\Gamma_{\propto+k}\Gamma_\beta}{\Gamma_{\propto+k+\beta}}\quad \left[\beta(\propto,\beta) = \frac{\Gamma_\propto \Gamma_\beta}{\Gamma_{\propto+\beta}}\right]$

$= \frac{\Gamma_{\propto+B}\ \Gamma_{\propto+k}}{\Gamma_\propto\ \Gamma_{k+\propto+\beta}}$

2) From the above formula , the mean and variance of the distribution can be derive as follows :- putting k=1 we obtain $E(x) = M_x = \frac{\Gamma_{\propto+1}\Gamma_{\propto+\beta}}{\Gamma_\propto \Gamma_\propto+\beta+1} = \frac{\propto\Gamma_\propto \Gamma_{\propto+\beta}}{\Gamma_\propto(\propto+\beta)\Gamma_{\propto+\beta}}\ [since\ \Gamma_{\propto+1} = \propto \Gamma_\propto]$

$M_x = \frac{\propto}{\propto+\beta}$

Putting k=2 we get $E(x^2) = \frac{\Gamma_{\propto+2}\Gamma_{\propto+\beta}}{\Gamma_\propto \Gamma_\propto+\beta+2} = \frac{(\propto+1)\propto\Gamma_\propto \Gamma_{\propto+}\beta}{\Gamma_\propto(\propto+\beta+1)(\propto+\beta)\Gamma_{\propto+\beta}}$

$= \frac{\propto(\propto+1)}{(\propto+\beta+1)(\propto+\beta)}$

Var (x) = $\int_x^2 = E(x^2) - [E(x)]^2$

$= \frac{\propto(\propto+1)}{(\propto+\beta+1)(\propto+\beta)} - \frac{\propto^2}{(\propto+\beta)^2} = \frac{\propto(\propto+1)(\propto+\beta)-\propto^2(\propto+\beta+1)}{(\propto+\beta)^2(\propto+\beta+1)}$

$= \frac{\propto^3+\propto^2\beta+\propto^2+\propto\beta-\propto^3-\propto^2\beta-\propto^2}{(\propto+\beta)^2(\propto+\beta+1)} = \frac{\propto\beta}{(\propto+\beta)^2(\propto+\beta+1)}$

$\therefore \int_x^2 = \frac{\propto\beta}{(\propto+\beta)^2(\propto+\beta+1)}$

5- The normal distribution

A continuous r.v x is said to have normal distribution with parameters M , δ^2 denote , as x~N(M, δ^2) if the p. d. f of x is $f(x) = \frac{1}{\sqrt{2\pi\ \delta^2}}\ e^{\frac{-1}{2}(\frac{x-M}{\delta})^2}$ $-\infty < x < \infty$

<u>Proportion :-</u>

1) The M. g. f of normal dist. Is $M_x(t) = e^{Mt\frac{\delta^2 t2}{2}}$

Proof :- $M_x(t) = E(e^{tx}) = \frac{1}{\sqrt{2\pi\delta^2}}\int_{-\infty}^{\infty} e^{+x}\ e^{\frac{-1}{2}}\ (\frac{x-M}{\delta})^2\ dx$

Iet y= $\frac{x-M}{\delta} \Rightarrow x = M + dy \Rightarrow dx = \delta \ dy$

$$M_x\,(t) = \frac{1}{\delta\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t(M+dy)} \quad e^{\frac{-y^2}{2}} \ (\delta \ dy)$$

$$= e^{Mt} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tdy} \quad e^{\frac{-y^2}{2}} \ dy$$

$$= e^{Mt} \ \frac{1}{\sqrt{2\pi}} \ \int_{-\infty}^{\infty} e^{\frac{-(y^2-2t\delta y)}{2}} dy$$

نجري بإكمال المربع على الاس بإضافة وطرح مربع نصف معامل y ينتج :

$$M_x \text{ (t)} = e^{\mu t} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-[(y^2-2\delta ty+\delta^2t^2)-\delta^2t^2]}{2}} dy$$

$$= e^{Mt} \ e^{\frac{\delta^2t^2}{2}} \ \frac{1}{\sqrt{2\pi}} \ \int_{-\infty}^{\infty} e^{\frac{-(y-\delta t)^2}{2}} dy$$

Let $Z = y - \delta t \Rightarrow y = Z + dt \Rightarrow dy = dZ$

$$M_x(t) = e^{Mt+\frac{\delta^2t^2}{2}} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-z^2}{2}} \ dZ\right]$$

normal dist $\therefore \ M_x(t) = e^{Mt+\frac{\delta^2t^2}{2}}$

2) **The mean of the normal dist is :-** $M_x = E(x) = M$

Proof : $M_x(t) = e^{Mt+\frac{\delta^2t^2}{2}}$

$$M_x(t) = (M + \frac{\delta^2t^2}{2}) \ e^{Mt+\frac{\delta^2t^2}{2}}$$

$=(M+\delta^2 t)M_x(t)$

$M_x = E(x) = M_x(0) = (M+0)M_x(0) = M.1 = M$

3) **The variance of the distribution is** $\delta x^2 = Var(x) = \delta^2$

Proof:-

$$M''_x(t) = (M+\delta^2 t)M'_{x(t)}\delta^2 M_x(t)[\text{since } M_x(t) = (M+\delta^2+)M_x(t)]$$

$$M''_x(0) = (M+0)M'_x(0) + \delta^2 M_x(0)$$

$= (M+0)\ M+\delta^2 - 1 = M^2 + \delta^2$

$\delta^2 = \text{var}(x) = M''_x(0) - [M'_x(0)]^2$

$= M^2 + \delta^2 - M^2 = \delta^2$

Def :- If the r.v $Z \sim N(0,1)$, then we say that Z distributed as standard normal distribution with p. d. f. $f(Z) = \frac{1}{\sqrt{2\pi}} e^{\frac{Z^2}{2}}$ $, -\infty < Z < \infty$

The mean , Variance and moment generating function of the r.v Z is

$M_Z = 0, \delta_{Z^2} = 1 , M_Z(t) = e^{t^2/2}$

Theorem 1) If the r.v $x \sim N(M, \delta^2)$then $Z = \frac{x-\mu}{\delta} \sim N\ (0,1)$

Proof :- By using the trans formation method we have f(x)=$\frac{1}{\sqrt{2\pi}} e^{\frac{-1}{2}(\frac{x-M}{\delta})^2}, -\infty < x < \infty$ the space of x denoted by A and the space of Z denoted by Bare defined as :-

A=$\{x = -\infty < x < \infty \}, \beta = \{Z = -\infty < Z < \infty\}$
$Z = u(x) = \frac{x-M}{\delta}\ is\ (1-1)$ trans formation maps A onto β
x=$u^{-1}(Z) = M + \delta Z\ is\ (1-1)$ transformation maps B onto A
$|J| = \left|\frac{dx}{dZ}\right| = d \implies g(Z) = f\ [u^{-1}(Z)]|J|$
$g(Z) = \frac{1}{\sqrt{2\pi}\delta} e^{\frac{-1}{2}Z^2} (\delta) = \frac{1}{\sqrt{2\pi}} e^{\frac{-1}{2}Z^2}$
$\therefore\ Z = \frac{x-M}{\delta} \sim N\ (0,1)$

Calculating the probabilities

The probabilities concerning the r.v.x which distributed as N (M, δ^2) can be expressed in terms of probabilities concerning $Z = \frac{x-M}{\delta}$ which distributed as N(0,1), However an integral like $\int_{-\infty}^{k} \frac{1}{\sqrt{2\pi}}\ e^{-Z2/2}\ \delta Z$ cannot be evaluated . Instead we use tables which approximate the value of this integral for differend values of k In general , the following rules are important .

1) $p_r(Z < 0) = p_r(Z > 0) = 0.5$
2) $p_r(Z < -Z_1) = 1 - p_r(Z < Z_1), Z_1 > 0$

3) $p_r(Z_1 < Z < Z_2) = p_r(Z < Z_2) - p_r(Z < Z_1) = N(Z_2) - N(Z_1)$

Ex:- Given that X~N (2,25) , find $p_r\ (0 < x < 10)$

Solution :- $p_r\ (0 < x < |0) = p_r(Z < \frac{10-2}{25})$

$= p_r\ (-0.4 < Z < 1.6)$

$= p_r\ (Z < 1.6) - p_r\ (Z < -0.4)$

$= p_r\ (Z < 1.6) - [1 - p_r\ (Z < 0.4)]$

$= N(1.6) - [1 - N(0.4)]$

= 0.45 -$[1 - 0.655] = 0.6$ [from tables]

Theorem (2) :- If the r.v X~N (M,δ^2) then the r.v y= $(\frac{x-M}{\delta})^2 \sim x^2(1)$

Proof :- By using the m. g. f. method $M_y(t) = E(e^{ty}) = E\left[e^{t(\frac{x-M}{\delta})^2}\right]$

$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\delta}\ e^{+(\frac{x-M}{\delta})^2}\ e^{\frac{-1}{2}(\frac{x-M}{\delta})^2}\ dx$

Putting $Z = \frac{x-M}{\delta} \Rightarrow x = M + \delta\, Z \Rightarrow dx = \delta dZ$

$M_x(t) = \frac{1}{\sqrt{2\pi}\,\delta} \int_{-\infty}^{\infty} e^{tZ^2 - \frac{1}{2}Z^2}(\delta d\,Z)$

$=\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{Z^2(t-\frac{1}{2})}\, dZ$

$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-Z^2}{2}(1-2t)} dZ$

Let w= $Z\sqrt{1-2t} \Rightarrow Z = \frac{1}{\sqrt{1-2t}}\, w\, , dZ = \frac{1}{\sqrt{1-2t}}\, dw$

$\Rightarrow M_y(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-w^2}{2}}\ \frac{1}{\sqrt{1-2t}}\ dw$

$= \frac{1}{\sqrt{1-2t}}\left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-w^2}{2}}\ dw\right] = \frac{1}{\sqrt{1-2t}}\ (1)$

$M_y(t) = (1-2t)^{-1/2}$ which is the m. g. f. of chi-square dist. with 1 degree of free

Doue y= $(\frac{x-\mu}{\delta})^2 \sim x^2(1)$

Theorem(3) :- If xi , i= 1,2,…,n distributed as $\sim N\ (0,1)$ then $\sum_{i=1}^{n} x_i^2 \sim x^2(n)$

Proof :- let y= $\sum_{i=1}^{n} x_i^2$, then $\mu_y(t) = E(e^{ty})$
$M_y(t) = E\left(e^{+(x_1^2+x_2^2+\cdots+x_n^2}\right)$

= E$(e^{tx_1^2})\ E\left(e^{tx_2^2}\right) \dots E(e^{tx_n^2})$

Since each of x_1, x_2,…,xn ~N(0,1) then each of $x_1^2, x_2^2, \dots, x_n^2\ \sim x^2(1)$ (theories 2)
$M_y(t) = (1-2t)^{-1/2}\,(1-2t)^{-1/2} \dots (1-2t)^{-1/2}$ n-terms
$M_y(t) = \left[((1-2t)^{-1/2}\right]^n = (1-2t)^{-n/2}$
y= $\sum_{i=1}^{n} xi^2 \sim x^2(n)$

(6) the students t distribution

Let the r.v W~N(0,1) and the r.v. V$\sim x^2(r)$, where W and V are stochastically independent . then T=$\frac{w}{\sqrt{\frac{v}{r}}}$ has students t distribution with p. d. f given by

$$g(t) = \frac{\Gamma_{(r+1)/2}}{\sqrt{\pi r}\,\Gamma_{\frac{r}{2}}} \quad \frac{1}{(1+\frac{t^2}{r})^{2\frac{(r+1)}{2}}} \quad , -\infty < t < \infty$$

Proof :- the joint p. d. f. of W and V is $\phi(w,v) = \frac{1}{\sqrt{2\pi}}\, e^{-w^2/2}\, \frac{1}{\Gamma_{\frac{r}{2}}\, 2^{r/2}}\, (v)^{\frac{r}{2}-1}\, e^{\frac{-v}{2}}$

$-\infty < w < \infty , 0 < v < \infty$

Let : $t = \frac{w}{\sqrt{\frac{v}{r}}}$ and u=v define $a(1-1)$ transformation mapping the space $\{(w,v)_i - \infty < w < \infty , 0 < v < \infty\}$ onto the space $\{(t,u)_i - \infty < t < \infty , 0 < u < \infty\}$

$$w = t\frac{\sqrt{u}}{\sqrt{r}} , v = u , J = \begin{vmatrix} \frac{dw}{dt} & \frac{dw}{du} \\ \frac{dv}{dt} & \frac{dv}{du} \end{vmatrix} = \begin{vmatrix} \frac{\sqrt{u}}{\sqrt{r}} & \frac{t}{2\sqrt{ur}} \\ 0 & 1 \end{vmatrix}$$

$$= \frac{\sqrt{u}}{\sqrt{r}}$$

$$g(t,u) = \emptyset \left[\frac{t\sqrt{u}}{\sqrt{r}} , u\right] \ . \ |J|$$