11) If $x \sim N(0,2)$ find $E(x^{k/2})$, where k is even positive number , then find $E(x^2)$

12) If the m.g.f of the r.v. x is $M_x(t) = e^{3t+8t^2}$ (i) find the distribution of x (ii) find the mean and variance of x

$s^2 \bar{x}$ تمارين

Problems :-

1- Let $\bar{x}$ be the mean of r.v of size(s) from N(0,12s) find the value of c if $p_r(\bar{x} < c) = 0.90$, and from tables it is known that N(1.282) = 0.90

2- Let $x_1, x_2, \dots, x_n$ be ar.s from G($\propto, \beta$) . show that $\bar{x} \sim G(\propto n, \frac{\beta}{n})$ then show that $E(\bar{x}) = \propto \beta , var\ (\bar{x}) = \frac{\propto \beta^2}{n}$

llint : $use\ M_{\bar{x}}(t) = E(e^{t\bar{x}}) = E(e^{\frac{t}{n}(x_1+x_2 \dots + x_n)})$

3- Let $x_1, x_2, \dots, x_n$ be ar.s from G(3,1) . find the p.d.f of $\bar{x}$, then find E($\bar{x}$) and var($\bar{x}$)

(4)Distribution of order statistics

Let $x_1, x_2, \dots, x_n$ denote ar,s from adist of continuous type having p.d.f (x) which is positive provided $a < x < b\ let\ y_1$ be the smallest of these x_i, y_2 the next x_i in order of magnitude , … and y_n the largest x_i , that is $y_1 < y_2 < \cdots < y_n$ represent $x_1, x_2, \dots, x_n$ when the latter are arranged in ascending order of magnitude , then $y_i, i = 1,2, \dots n$ is called the i^{th} order statistic of the r.s $x_1, x_2, \dots, x_n$

It can be shown that the joint p.d.f of $y_1, y_2, \dots, y_n$ is $g(y_1, y_2, \dots, y_n) = n!\, f(y_1) f(y_2) \dots f(y_n) \dots . (1)$ the marginal distribution of y_k is g(y_k) = $\frac{n!}{(k-1)!(n-k)!}\ [f(y_k)]^{k-1}\ [1 - f(y_k)]^{n-k}\ f(y_k) \dots . (2)$

The joint p.d.f for any two order statistics y_i, y_j is $g(y_i, y_j) = \frac{n!}{(i,1)!(j-i-1)!(n-j)!} [f(y_i)]^{i-1}[f(y_j) - f(y_i)]^{j-i-1}$
$[1 - f(y_j)]^{n-j} f(y_i) f(y_j) \dots (3)$

Ex:- let $y_1 < y_2 < y_3 < y_4$ denote the order statist ion of ar.s of size u from a distribution having p.d.f $f(x) = \begin{cases} 2x & 0<x<1 \\ 0 & o.w \end{cases}$

Find the p.d.f of y_3 then find $p_r(\frac{1}{2} < y_3)$

Solution :- Applying formula (2) with n=4 and $k = 3$ we get $g(y_3) = \frac{u!}{2! \quad 1!} [f(y_3)]^2 [1 - f(y_3)]^2 f(y_3)$

Since f(x)= $\int_0^x f(u)du = \int_0^x 2udu = \frac{2u^2}{2} \Big|_0^x = x^2$ then f(y_3) $= {y_3}^2$
$g(y_3) = 12[{y_3}^2]^2 [1 - {y_3}^2] 2y_3$

= 24 ${y_3}^4 (1 - {y_3}^2) y_3 = 24 {y_3}^5 (1 - {y_3}^2) \quad 0 < y_3 < 1$
$p_r\left(\frac{1}{2} < y_3\right) = \int_{1/2}^1 24\, {y_3}^s\, (1 - {y_3}^2) d\, y_3$

$= \int_{1/2}^1 24\, ({y_3}^s - {y_3}^7) d\, y_3 = 24[\frac{{y_3}^6}{6} - \frac{{y_3}^8}{8} \Big|_{1/2}^1$

Distribution of functions of order statistic :-

(1) The median : when the observations are arranged in ascending order of magnitude , then (a) if n is odd , the median is the observation of orders $\frac{n+1}{2}$ (b) if n is even , the median is the average observation of order $\frac{n}{2}, \frac{n}{2} + 1$ if x is ar.v with p.d.f f(∝) and distribution function f(∝), then the value of the median is the value of x that satisfies the equation f$(x) = \frac{1}{2}$

Ex:- let x_1, x_2, x_3 be ar.s from the dist. $f(x) = e^{-x}, x > 0$

(i) find the p.d.f of the smallest value of the sample
(ii) find the joint p.d.f of the largest and smallest value of the sample
(iii) find the p.d.f of the median and the value of the median

solution : let y_1 is the smallest value of the sample followed by y_2 the y_3
$f(x) = e^{-x} \Rightarrow f(x) = \int_0^x f(u)du = \int_0^x e^{-u} du = -e^{-u} \Big|_0^x$

$= \ -e^{-x} + e^{\infty} = 1 - e^{-x}$ from formula (2) with $k = 1$ and n=3 we get $g(y_k) = \frac{n!}{(|c-1)!(n-|c)!} \ [f(y_k)]^{k-1}\big(1 - f(y_k)\big)^{n-k} f(y_k)$

$g(y_1) = \frac{3!}{0! \ \ 2!} \quad [f(y_1)]^0 \ [1 - f(y_1)]^2 \ f(y_1)$

$g(y_1) = 3 \ (1 - 1 + e^{-y_1})^2 e^{-y_1} = 3 \ e^{-2y_!} \ e^{-y_1} \ = 3e^{-3y_1}$

(ii) from formula (3) with i=1, j=3 , n=3 $g(y_1, y_3) = \frac{3!}{0! \ 1! \ 0!} \ [f(y_1)]^0 [f(y_s) \quad f(y_1)]^{3-1-1} [1 - f(y_3)]^0 f(y_1) f(y_3)$

= 6[1-$e^{-y_3} - (1 - e^{-y_1})]e^{-y_1} e^{-y_3}$

= 6[x-$e^{-y_3} - y + e^{-y_1}]e^{-y_1} e^{-y_3}$

= 6(-$e^{-y_3} + e^{-y_1})e^{-y_1} e^{-y_3} = \ 6(e^{-y_3} + e^{-y_1})e^{-(y_1+y_3)} \quad 0 < y_1 < y_3 < \infty$

(iii)the median is the observation of order $\frac{n+1}{2} = \frac{3+1}{2} = 2$ which is y_2
g(y_2) $= \frac{3!}{1! \ \ 1!} \ f(y_2)[1 - f(y_2)^{-(1-e^{-y_2})} \ ] \ f(y_2)$

= 6$(1 - e^{-y_2})e^{y_2 e^{-y_2}} = 6 \ e^{-2y_2}(1 - e^{-y_2}) \qquad 0 < y_2 < \infty$

To find the value of y_2 we get $f(y_2) = \frac{1}{2} \Rightarrow 1 - e^{-y_2} = \frac{1}{2} \Rightarrow e^{-y_2} = \frac{1}{2} \qquad y_2 = ln_2$

The Range :- the range of the sample is the difference between the largest and smallest valne of the sample that is $R = y_n - y_1$

Ex:- let x_1, x_2, x_3 be ar.s from B(2,1) and let $y_1 < y_2 < y_3$ be the order statistics of the sample

(a) find the probability distribution of R
(b) find the mean and variance of the distribution

solution :- $x \sim B(2^{\propto}, 1^{B})$

$f(x, \propto, B) = \frac{\Gamma_{\propto+B}}{\Gamma_{\propto}\Gamma_{B}} \ x^{\propto-1}(1 - x)^{B-1} \qquad 0 < x < 1$

if $\propto = 2$, $B = 1$ then $f(x) = \frac{\Gamma_3}{\Gamma_1\Gamma_2} \ \ x(1 - x)^0$

= 2x $\qquad 0 < x < 1$

$f(x) = \ \int_0^x f(u)du = \ \int_0^x 2udu = \frac{2u^2}{2} \ \ \Big|_0^x = \ x^2$

Let $\left.\begin{matrix} R = y_3 - y_1 = u_1(y_1, y_3) \\ Z = y_3 = u_2(y_1, y_2) \end{matrix}\right\}$(1-1) trans formation from space of y_1, y_3 to space of R Z

$\left.\begin{matrix} y_1 = Z - R = u_1{}^{-1}(R, Z) \\ y_3 = Z = u_2{}^{-1}(RZ) \end{matrix}\right\}$ (1-1) transformation from space of R, Z to space of y_1, y_3

Applying formula (3) we get $g(\ y_1, y_3) = 6[f(y_3) - f(\ y_1)]f(\ y_1)f(y_3)$

$= 6\ [y_3{}^2 - y_1{}^2] \quad (2y_1)\ (2y_3)$

$= \quad 24 \quad y_1 y_3 \quad [y_3{}^2 - y_1{}^2] \qquad 0 < y_1 < y_3 < 1$

$J = \begin{vmatrix} \frac{dy_1}{dZ} & \frac{dy_1}{dR} \\ \frac{dy_3}{dZ} & \frac{dy_3}{dR} \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} = 1$

$g(R, Z) = g[u_1{}^{-1}(R, Z), u_2{}^{-1}(R, Z)]\ |J|$

$=24[Z^2 - (Z-R)^2](Z-R)Z] \qquad 0 < R < 8 < 1$

$= 24\ [Z^2 - Z^2 + 2RZ - R^2] \quad (Z-R)Z$

$= 24\ [2Z^2R - R^2Z]\ (Z-R)$

$= 24\ [2Z^3R - R^2Z^2 - 2Z^2R^2 + R^3Z]$

$= 24\ R\ [2Z^3 - R^2Z^2 - 2Z^2R + R^2Z]$

$=24R \qquad [2Z^3 - 3Z^2R + R^2Z]$

$h\ (R) = \int_R^1 g(R, Z)dZ = \int_R^1 24R\ [2Z^3 - \ 3Z^2R + R^2Z]dZ$

$= 24R(\frac{2Z^4}{4} - \frac{2Z^3R}{3} + \frac{R^2Z^2}{2}\ \Big|_R^1$

$= 24R\ (\frac{1}{2} - R + \frac{R^2}{2} - \frac{1}{2}\ R^4 + R^4 - \frac{R^4}{2}$

$= \qquad \frac{24}{2}\ R\ \begin{matrix} [1-2R+R^2] = 12R(R-1)^2 \\ [R^2-2R+1] = 12\ (1-R)^2R \end{matrix}$

$R \sim B\ (2,3)$

$E\ (R) = \frac{\propto}{\propto + B} = \frac{2}{5}$

$Var\ (R) = \frac{\propto B}{(\propto + B)^2(\propto + B + 1)} = \frac{6}{25(6)} = \frac{1}{25}$

Ex:- let $y_1 < y_2 < \cdots < y_s$ denote the order statistics of ar.s. of size 5 from a distribution having p.d.f. $f(x) = e^{-x}, 0 < x < \infty$ show that the statistics $Z_1 = y_2, Z_2 = y_4 - y_2$ are stochastically in dependent .

Solution :- $F(x) = 1 - e^{-x}, 0 < x < \infty$ and according to formula (2) we get . $g(y_2, y_4) = \frac{5!}{1!\ \ 1!\ \ !1}(1 - e^{-y_2})[1 - e^{-y_4} - (1 - e^{-y_2})]$
$[1\text{-}(1\text{-}e^{-y_4})]e^{-y_2 - y_4}$

$=5!\,(1 - e^{-y_2})[e^{-y_2} - e^{-y_4}]e^{-y_4}\ e^{-y_2-y_4} \qquad 0 < y_2 < y_4 < \infty$

Let the space of y_2, y_4 is $A = \{(\,y_2, y_4): 0 < y_2 < y_4 < \infty\,\}$ the space of Z_1, Z_2 is $B = \{(Z_1, Z_2): 0 < Z_1 < \infty, 0 < Z_2 < \infty\}$

$\left.\begin{matrix} Z_1 = y_2 = u_1(y_1, y_2) \\ Z_2 = y_4 - y_2 = u_2(y_1, y_2) \end{matrix}\right]$ (1-1) trans formation maps A onto B

$\left.\begin{matrix} y_2 = Z_1 = u_1^{-1}(Z_1, Z_2) \\ y_4 = Z_2 + Z_1 = u_2^{-1}(Z_1, Z_2) \end{matrix}\right]$ (1-1) transformation maps B onto A

$$J = \begin{vmatrix} \frac{\partial y_2}{\partial Z_1} & \frac{\partial y_2}{\partial Z_2} \\ \frac{\partial y_4}{\partial Z_1} & \frac{\partial y_4}{\partial Z_2} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1$$

$$g(Z_1, Z_2) = g(u_1^{-1}(Z_1, Z_2), u_2^{-1}((Z_1, Z_2)\ |J|$$

$$= 120\ (1\text{-}e^{-Z_1})\left(e^{-Z_1} - e^{-(Z_2+Z_1)}\right)e^{-(Z_2+Z_1)},\ e^{-Z_1 - Z_2 - Z_1}$$

$$= 120\ (1\text{-}e^{-Z_1})\ e^{-Z_1}(1 - e^{-Z_2})\ e^{-Z_2}e^{-Z_1}e^{-2Z_1}e^{-Z_2}$$

$$= 120\ e^{-4Z_1}(1 - e^{-Z_1})e^{-2Z_1}(1 - e^{-Z_2}) \qquad ,0 < Z_1 < \infty\ \ ,\ 0 < Z_2 < \infty$$

The marginal distribution for each of Z_1, Z_2 are as follows :- $h(Z_1) = \int_0^\infty g(Z_1, Z_2)d\,Z_2 = \int_0^\infty 120\ e^{-4Z_1}(1 - e^{-Z_1})\ e^{-2Z_2}(1 - e^{-Z_2})dZ_2$

$$= 120\ e^{-4Z_1}(1 - e^{-Z_1})\ \int_0^\infty e^{-2Z_2}(1 - e^{-Z_2})dZ_2$$

$$= 120\ e^{-4Z_1}\ (1 - e^{-Z_1})\int_0^\infty(e^{-2Z_2} - e^{-3Z_2})\ dZ_2$$

$$= 120\ \ e^{-4Z_1}\ (1 - e^{-Z_1})\ (\frac{-1}{2}\ e^{-2Z_2} + \frac{1}{3}\ e^{-3Z_2}\Big|_0^\infty$$

$$= 120\ \ e^{-4Z_1}\ (1 - e^{-Z_1})\ (\frac{1}{2} - \frac{1}{3})$$

$= 120\ e^{-4Z_1}\ (1 - e^{-Z_1})\ (\frac{3-2}{6})$

$=$ 20 e^{-4Z_1} $(1 - e^{-Z_1})$

$h(Z_2) = \int_0^\infty g(Z_1, Z_2) dZ_1$

$= 120\ e^{-2Z_2}\ (1 - e^{-Z_2}) \int_0^\infty e^{-4Z_1}\ (1 - e^{-Z_1}) dZ_1$

$= 120\ e^{-2Z_2} (1 - e^{-Z_2}) \int_0^\infty e^{-4Z_1}\ - e^{-sZ_1} dZ_1$

$= 120\ e^{-2Z_2} (1 - e^{-Z_2})\ (\frac{-1}{4} e^{-4Z_1} + \frac{1}{5} e^{-5Z_1} \Big|_0^\infty$

$= 120\ e^{-2Z_2} (1 - e^{-Z_2})\ (\frac{1}{4} - \frac{1}{5})$

$= 120\ e^{-2Z_2} (1 - e^{-Z_2})\ \frac{5-4}{20}$

$= 6e^{-2Z_2} (1 - e^{-Z_2})$

Since $g(Z_1, Z_2) = h(Z_1) h(Z_2)$ then Z_1, Z_2 are stocuastically independent .

Problems :-

(1) let $y_1 < y_2 < y_3 < y_4$ be the orde. Statistics of r.s of size 4 from the dist having p.d.f f(x)=e^{-x}, $0 < x < \infty$, final $p_r(s \leq y_4)$

(2) let x_1, x_2, x_3 be ar.s from f(x)=2x , $0 < x < 1$, compute the probability that the smallest of these xi excecds the median of the distribution .