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Estimation Local Fractional L_p Normed Spaces for $1 \le p < \infty$

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A B S T R A C T

This paper introduces an improvement and generalization of Hadamard inequality by using the spaces of local fractional L_p-normed spaces for $1 \le p < \infty$, after Jacques Hadamard has presented his inequality in 1893 about bound on the determination of matrices by using the length of its column vectors. In addition, it is sometimes called Hemit Hadamard inequality with its convex function .The inequality that had been generalized to higher dimensions of function is bounded and convex domain in Euclidean space of n dimension .

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1.Introduction

Hadmard introduced the first type of Hermite-Hadmard inequality (Hadmard,j,1893)

$$\varphi\left(\frac{\vartheta+\omega}{2}\right) \le \frac{1}{\omega-\vartheta}\int_{\vartheta}^{\omega}\varphi(x)dx \le \frac{\varphi(\vartheta)+\varphi(\omega)}{2} \qquad (1)$$

Bullen presented a generalization of Hermite-Hadmard estimate and proved (Bullen ,P,S,1978)

$$\frac{1}{\omega-\vartheta}\int_{\vartheta}^{\omega}\varphi(x)dx \le \frac{1}{2}\left[\varphi\left(\frac{\vartheta+\omega}{2}\right) + \frac{\varphi(\vartheta)+\varphi(\omega)}{2}\right] \qquad (2)$$

you can look at a general survey for Bullen inequality in (Bullen ,P,S,1978)

, see also(Mitrinovic ,D.S.Pecaric ,J and Pecaric ,Jand Fink ,A.M.2013) and (Dragomir,S.S and Agarwal,R.P,1998).

As a refinement of (1), is the following result from (El Farissi, A,2010)

Theorem : Adopted $\varphi: I \to R$ is a curved role I. At that point for all λϵ[0,1],we will have

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$$\varphi\left(\frac{\vartheta+\omega}{2}\right) \le I(\lambda) \le \frac{1}{\omega-\vartheta}\int_{\vartheta}^{\omega}\varphi(x)dx \le L(\lambda)\frac{\varphi(\vartheta)+\varphi(\omega)}{2}$$

Where

$$I(\lambda) = \lambda\varphi\left(\frac{\lambda\omega+(1-\lambda)\vartheta}{2}\right) + (1-\lambda)\varphi\left(\frac{(1+\lambda)\omega+(1-\lambda)\vartheta}{2}\right).$$

And

L(ʎ)=1/2(φ(ʎω+(1-ʎ)ϑ)+ʎφ(ϑ)+(1-ʎ)φ(ω))

To read more about the inequalities mentioned above, we can read the references [Mitrinovic ,D.S.Pecaric ,J and Pecaric ,Jand Fink ,A.M.2013), (El Farissi, A,2010) , (Meftah,B & Boukerrioua Khaled 2017),(Meftah,B, M. Merad & A. Souahi 2019), (Eman ,S ,1999)]

This paper introduces an improvement and generalization of Hermite- Hadamard version inequalities and applications on the local fractional L_p normed spaces for $1 \le p < \infty$.

Firstly let us recall and offer some definitions and notations, which are required in this study.

The function $\varphi: [\vartheta,\omega] \subset R \to R$ is supposed to be seen curved if the subsequent dissimilarity clutches

$$\varphi(tx+(1-t)y \le t\varphi(x)+(1-t)\varphi(y)$$

(Mitrinovic ,D.S.Pecaric ,J and Pecaric ,Jand Fink ,A.M.2013)

For all $x,y \in [\vartheta,\omega]$ and $t \in [0,1]$, the researchers declare that is curved in if $(-\varphi)$ is curved.

R^{α} is the set of real numbers, $R^{\alpha} = Q^{\alpha} \cup J^{\alpha}$,where Q^{α} is the $\alpha-$ kind fixed set of the rational number accounts is demarcated as the group

$$\left\{m^{\alpha} = \left(\frac{p}{q}\right)^{\alpha}, p, q\epsilon Z, q \neq 0\right\},$$

In addition, J^{α} is the $\alpha-$ the type set of irrational numbers is clarified as the fixed set.

$$\left\{m^{\alpha} \neq \left(\frac{p}{q}\right)^{\alpha}, p, q\epsilon Z, q \neq 0\right\}.$$

The local fractional derivative of the function $\varphi(x)$ of demand α at $X = X_\circ$ is known as:

$$\varphi^{\alpha}(X_\circ) = \left.\frac{d^{\alpha}\varphi(x)}{dx^{\alpha}}\right|_{X=X_\circ} = \lim_{X\to X_\circ}\frac{\Delta^{\alpha}(\varphi(X)-\varphi(X_\circ))}{(X-X_\circ)}$$

Where $\Delta^{\alpha}(\varphi(X)-\varphi(X_\circ)) \cong \Gamma(\alpha+1)\big(\varphi(X)-\varphi(X_\circ)\big)$.

If there occurs $\varphi^{(k+1)\alpha}(x) = D_x^{\alpha} \dots D_x^{\alpha}\varphi(x)$ for any $X \in I \subseteq R$, then we signified $\varphi \in D_{(k+1)\alpha}(I)$, wherever $k = 0,1,\dots$ (Dragomir,S.S and Agarwal,R.P,1998)

Anon –differentiable function $\varphi: R \to R^{\alpha}, X \to f(x)$ is termed as fractional locally continues at $X_\circ$. "If for whichever $\epsilon > 0$, there be existent $\delta > 0$, as that $|\varphi(x)-\varphi(X_\circ)| < \epsilon^{\alpha}$ holds for $|X - X_\circ| < \delta$, where $\epsilon, \delta \in R$. If $\varphi(x)$ is local incessant on the interlude (ϑ,ω) .We represent $\varphi(x) \in C_{\alpha}(\vartheta,\omega)$". (Dragomir,S.S and Agarwal,R.P,1998)

I call f is fractional integral if

$$\frac{1}{\Gamma(1+\alpha)}\int_{\vartheta}^{\omega}\varphi(x)(dt)^{\alpha} = \frac{1}{\Gamma(1+\alpha)}\lim_{\Delta t\to 0}\sum_{j=0}^{n-1}\varphi(t_j)\left(\Delta t_j\right)^{\alpha} < \infty \quad (3)$$

and the fractional integral is defined by:

$$\vartheta I_{\omega}^{\alpha} f(x) = \frac{1}{\Gamma(1+\alpha)}\int_{a}^{\omega}\varphi(x)(dt)^{\alpha} = \frac{1}{\Gamma(1+\alpha)}\lim_{\Delta t\to 0}\sum_{j=0}^{n-1}\varphi(t_j)\left(\Delta t_j\right)^{\alpha}$$

With $\Delta t_j = t_{j+1} - t$,and $\Delta t = max\{\Delta t_1, \Delta t_2, \dots, \Delta t_{n-1}\}$,where $[t_j, t_{j+1}]$, $j = 0,\dots,n-1$, and $\vartheta = t_0 < t_1 < \cdots < t_{n-1} = \omega$ is a partition of interval $[a,b]$. Here it follows that $\vartheta I_{\omega}^{\alpha}\varphi(x) = 0$ if $\vartheta = \omega$ and $\vartheta I_{\omega}^{\alpha}\varphi(x) = -\omega\, \vartheta I_{\omega}^{\alpha}\varphi(x)$ if $\vartheta < \omega$.

Let $\varphi: I \subset R \to R^{\alpha}$. As such, any $X_1, X_2 \in I$and $\lambda \in [0,1]$, if the subsequent inequity $\varphi(\lambda X_1) + (1-\lambda)X_2 \le \lambda^{\alpha} f(X_1) + (1-\lambda)^{\alpha}\varphi(X_2)$ grasps, then f is named a widespread convex role on I.

Now, let us introduce our $L_{P,\propto}$ space for $0 < p < \infty$.

Let us define the fractional integral quasi normed space as:

$$L_{P,\propto}[\vartheta,\omega] = \left\{\varphi:[\vartheta,\omega] \to R: \|\varphi\|_{P,\alpha} = \left(\int_{\vartheta}^{\omega} |\varphi(x)|^p \, (dx)^\alpha\right)^{\frac{1}{P}} < \infty\right\}$$

and $\|.\|_{P,\alpha}$ is a fractional L_P integrable norm.

2.Auxilary Results

Lemma 2.1 (Barnett,N.S. and Dragomir,S.S,2001):

$$\frac{d^\alpha f(x)}{dx^\alpha} = \frac{\Gamma(1+k\alpha)}{\Gamma(1+(k-1)\alpha)} X^{(k-1)\alpha}$$

$$\frac{1}{\Gamma(1+\alpha)}\int_{\vartheta}^{\omega} X^{k\alpha}(dx)^\alpha = \frac{\Gamma(1+k\alpha)}{\Gamma(1+(k+1)\alpha)}\left(\omega^{(k+1)\alpha} - \vartheta^{(k+1)\alpha}\right), k\epsilon R.$$

Lemma 2.2: (Meftah & Boukerrioua Khaled ,2017) GeneraliZed Holder inequality

Let φ', g $\in C_\alpha[\vartheta,\omega], p,q > 1$, with $\frac{1}{p}+\frac{1}{q}=1$, then

$$\frac{1}{\Gamma(1+\alpha)}\int_{\vartheta}^{\omega} |\varphi(x)g(x)|(dx)^\alpha \le \left(\frac{1}{\Gamma(1+\alpha)}\int_{\vartheta}^{\omega}|\varphi(x)|^p(dx)^\alpha\right)^{\frac{1}{p}} \cdot \left(\frac{1}{\Gamma(1+\alpha)}\int_{\vartheta}^{\omega}|g(x)|^q(dx)^\alpha\right)^{\frac{1}{q}}.$$

Lemma 2.3 (Dragomir ,s.s and Pearce ,C.E.M(2000):Let $\varphi \in D_\alpha[\vartheta,\omega]$ with $\vartheta < \omega$ and $g(x) \in C_\alpha[\vartheta,\omega]$.If $\varphi^\alpha \in I_x^{(\alpha)}[\vartheta,\omega]$,then for all $x \in [\vartheta,\omega]$,the following indentity holds.

$$"\varphi(\vartheta)\vartheta I^{(\propto)}{}_x g(s) + \varphi(\omega) x I^{(\propto)}{}_\omega g(s) - \vartheta I^{(\propto)}{}_\omega \varphi(s) g = \frac{1}{\Gamma(1+\alpha)}\int_{\vartheta}^{\omega}\left[\frac{1}{\Gamma(1+\alpha)}\int_x^t g(s)(ds)^\alpha\right]\boldsymbol{\varphi}^{(\propto)}(t)(dt)^\alpha"$$

3.Main Results

Theorem 3.1:

let $\varphi(x) \in L_{p,\alpha}[\vartheta,\omega]$ and $\varphi^\propto$ is generalized convex in $L_p[\vartheta,\omega]$.

Then

$$\left|\varphi(\vartheta)_a I^{(\propto)}{}_x g(s) + \varphi(\omega)_x I^{(\propto)}{}_\omega g(s) - \vartheta I^{(\propto)}{}_\omega \varphi(s) g(s)\right| xs \le \frac{((x-t)^\alpha)^{\frac{1}{p}}}{\Gamma(1+\alpha)^{p+q+1}} \cdot \frac{\Gamma(1+\alpha q)}{\Gamma(1+(q+1)\alpha)}\left(\omega^{(q+1)\propto} - \vartheta^{(q+1)\propto}\right)\|g(s)\|_{p,\propto}$$

$$where \qquad 1 \le p,q \le \infty, \frac{1}{p}+\frac{1}{q} = 1.$$

Proof: By using lemma2.3, we get

$$"\left|\varphi(\vartheta)_\vartheta I^{(\propto)}{}_x g(s) + \varphi(\omega)_x I^{(\propto)}{}_\omega g(s) - \vartheta I^{(\propto)}{}_\omega \varphi(s) g(s)\right|$$

$$= \left|\frac{1}{\Gamma(1+\alpha)}\int_{\vartheta}^{\omega}\left[\frac{1}{\Gamma(1+\alpha)}\int_x^t g(s)(ds)^\alpha\right]\boldsymbol{\varphi}^{(\propto)}(t)(dt)^\alpha\right|$$

Currently via employing general Holders inequality in lemma2.2, we gain

$$\left|\frac{1}{\Gamma(1+\alpha)}\int_{\vartheta}^{\omega}\left[\frac{1}{\Gamma(1+\alpha)}\int_x^t g(s)(ds)^\alpha\right]\boldsymbol{\varphi}^{(\propto)}(t)(dt)^\alpha\right| \le$$

$$\frac{1}{\Gamma(1+\alpha)}\left\{\left[\int_{\vartheta}^{\omega}\left[\frac{1}{\Gamma(1+\alpha)}\int_x^t |g(s)|^p(ds)^\alpha\right]\right]^{\frac{1}{p}}\left[\int_{\vartheta}^{b}\left|\varphi^{(\propto)}(t)\right|^q(dt)^\alpha\right]^{\frac{1}{q}}\right\} 1 \le p,q \le \infty, \frac{1}{p}+\frac{1}{q} = 1.$$

$$= \frac{1}{\Gamma(1+\alpha)}(I_1 + I_2)$$

Now,

$$I_1 = \left[\left|\int_{\vartheta}^{\varphi}\left[\frac{1}{\Gamma(1+\alpha)}\int_x^t |g(s)|^p (ds)^\alpha\right]\right|\right]^{\frac{1}{p}}$$

$$\leq \left[\left|\int_{\vartheta}^{\omega} |g(s)(ds)|^p \,(dt)^\alpha\right|\left|\frac{1}{\Gamma(1+\alpha)}\int_x^t (ds)^\alpha\right|\right]^{\frac{1}{p}}$$

by using lemma 2.1 and (1), we get,

$$I_1 = \left(\left|\frac{1}{\Gamma(1+\alpha)}\int_x^t |g(s)(ds)|^p (dt)^\alpha\right|\right)^{\frac{1}{p}}$$

$$\leq \|g(s)\|_{p,\propto}\left(\frac{(x-t)^\alpha}{\Gamma(1+\alpha)}\right)^{\frac{1}{p}} . \qquad (3)$$

By using lemma 2.1,we get,

$$I_2 = \left[\int_{\vartheta}^{\omega}\left|\varphi^{(\propto)}(t)\right|^q (dt)^\alpha\right]^{\frac{1}{q}}$$

$$= \frac{\Gamma(1+\alpha q)}{[\Gamma(1+\alpha)]^{\frac{1}{q}}\Gamma(1+(q+1)\alpha)}\left(\omega^{(q+1)\propto} - \vartheta^{(q+1)\propto}\right). \qquad (4)$$

Put (3) and (4) in (2), we get

$$\frac{1}{\Gamma(1+\alpha)}\left\{\left[\int_{\vartheta}^{\omega}\left[\frac{1}{\Gamma(1+\alpha)}\int_x^t |g(s)|^p (ds)^\alpha\right]\right]^{\frac{1}{p}}\left[\int_{\vartheta}^{b}\left|\varphi^{(\propto)}(t)\right|^q (dt)^\alpha\right]^{\frac{1}{q}}\right\}$$

$$= \frac{((x-t)^\alpha)^{\frac{1}{p}}}{\Gamma(1+\alpha)^{p+q+1}} \cdot \frac{\Gamma(1+\alpha q)}{\Gamma(1+(q+1)\alpha)}\left(\omega^{(q+1)\propto} - \vartheta^{(q+1)\propto}\right)\|g(s)\|_{p,\propto}$$

Now using (1),we catch

$$\left|\varphi(\vartheta)_a I^{(\propto)}{}_x g(s) + \varphi(\omega)_x I^{(\propto)}{}_\omega g(s) - \vartheta I^{(\propto)}{}_\omega \varphi(s) g(s)\right|$$

$$\leq \frac{((x-t)^\alpha)^{\frac{1}{p}}}{\Gamma(1+\alpha)^{p+q+1}} \cdot \frac{\Gamma(1+\alpha q)}{\Gamma(1+(q+1)\alpha)}\left(\omega^{(q+1)\propto} - \vartheta^{(q+1)\propto}\right)\|g(s)\|_{p,\propto}$$

$$where \qquad 1 \leq p,q \leq \infty, \frac{1}{p}+\frac{1}{q} = 1.$$

Also,

$$I_2 = \left[\int_{\vartheta}^{\omega}\left|\varphi^{(\propto)}(t)\right|^q (dt)^\alpha\right]^{\frac{1}{q}}$$

$$\leq [\Gamma(1+\alpha)]^{\frac{1}{q}}\left[\frac{1}{\Gamma(1+\alpha)}\int_{\vartheta}^{\omega}\left|\varphi^{(\propto)}(t)\right|^q (dt)^\alpha\right]^{\frac{1}{q}}.$$

Corollary 3.2:

Let $g:[\vartheta,\omega] \to R^\alpha$ be a asymmetric to $(\vartheta+\omega)/2$. Then we have the a new Hadamard inequality.

$$"\left|\frac{(\varphi(\vartheta)+\varphi(\omega))}{2^\alpha} - \vartheta I_\omega^\propto g(s) - \vartheta I_\omega^\propto \omega(s) g(s)\right|$$

$$\leq \frac{\Gamma(1+\alpha q)\left(\left(\frac{\vartheta+\omega}{2}-t\right)^\propto\right)^{\frac{1}{p}}\|g(s)\|_{p,\propto}}{\left(\Gamma(1+(q+1)\alpha)\right)^{\frac{1}{q}}\left(\Gamma(1+\alpha)\right)^{\frac{1}{p+q+1}}}\left(b^{(q+1)\propto} - \vartheta^{(q+1)\propto}\right),$$

$$where \qquad 1 \leq p,q \leq \infty, \frac{1}{p}+\frac{1}{q} = 1."$$

Proof:

Put $x = \frac{\vartheta+\omega}{2}$ in theorem3.1, we get

$$\left|\varphi(\vartheta)_a I^{(\propto)}{}_{\frac{\vartheta+\omega}{2}} g(s) + \varphi(b)_{\frac{\vartheta+\omega}{2}} I^{(\propto)}{}_\omega g(s) - \vartheta_\omega \varphi(s) g(s)\right|$$

$$\leq \frac{\left(\left(\frac{\vartheta+\omega}{2}-t\right)^\alpha\right)^{\frac{1}{p}}}{\Gamma(1+\alpha)^{p+q+1}} \cdot \frac{\Gamma(1+\alpha q)}{\Gamma(1+(q+1)\alpha)}\left(\omega^{(q+1)\propto} - \vartheta^{(q+1)\propto}\right)\|g(s)\|_{p,\propto} \qquad (5)$$

Now,

using the symmetry of g , we have the following identity

$$\varphi(\vartheta)_{\vartheta}I^{(\propto)}{}_{\frac{\vartheta+\omega}{2}}g(s)+\varphi(b)_{\frac{\vartheta+\omega}{2}}I^{(\propto)}{}_{b}g(s) - \vartheta I^{(\propto)}{}_{\omega}\varphi(s)g(s) = \frac{\varphi(\vartheta)+\varphi(\omega)}{2^{\alpha}} - \vartheta I_{\omega}^{\propto}g(s) - \vartheta I_{\omega}^{\propto}\varphi(s)g(s) \quad (6)$$

Put (6) in (5), we get

$$\left|\frac{\varphi(\vartheta)+\varphi(\omega)}{2^{\alpha}} - \vartheta I_{\omega}^{\propto}g(s) - \vartheta I_{\omega}^{\propto}\varphi(s)g(s)\right|$$

$$\leq \frac{\Gamma(1+\alpha q)\left(\left(\frac{\vartheta+\omega}{2}-t\right)^{\propto}\right)^{\frac{1}{p}}\|g(s)\|_{p,\propto}}{\left(\Gamma(1+(q+1)\alpha)\right)^{\frac{1}{q}}\left(\Gamma(1+\alpha)\right)^{\frac{1}{p+q+1}}}\left(\omega^{(q+1)\propto} - \vartheta^{(q+1)\propto}\right).$$

Theorem 3.3: *Let* $f(x) \in D_{\propto}[\vartheta, b]$ *with* $\vartheta < \omega$ *and let* $\varphi^{(\propto)} \in I_x^{(\alpha)}[\vartheta,\omega]$, $g^{(\propto)}(x) \in C_{\alpha}[\vartheta,\omega]$.

If $\left|\varphi^{(\propto)}\right|^{p/(p-1)}$ is generalized convex on [ϑ,ω] with p>1, then for all $x \in [\vartheta,\omega]$, the succeeding inequality grips

"$$\left|\varphi(\vartheta)_{a}I^{(\propto)}{}_{x}g(s)+\varphi(\omega)_{x}I^{(\propto)}{}_{\omega}g(s) - \vartheta I^{(\propto)}{}_{\omega}\varphi(s)g(s)\right| \leq \frac{((x-t)^{\alpha})^{\frac{1}{p}}}{\Gamma(1+\alpha)^{\frac{p^2+p-1}{p-1}}} \cdot \frac{\Gamma\left(\frac{p-\propto+\alpha p}{p}\right)}{\Gamma\left(\frac{p-\propto+2\alpha p}{p}\right)}\left(\omega^{\frac{2\propto p-\propto}{p}} - \vartheta^{\frac{2\propto p-\propto}{p}}\right)\|g(s)\|_{p,\propto}$$

$$where \quad 1 \leq p,q \leq \infty, \frac{1}{p}+\frac{1}{q}=1".$$

Proof: Let $x \in [\vartheta,\omega]$, by using lemma 2.3 for $\varphi^{(\propto)}$, we get

$$\left|\varphi(\vartheta)_{\vartheta}I^{(\propto)}{}_{x}g(s)+\varphi(\omega)_{x}I^{(\propto)}{}_{\omega}g(s) - \vartheta I^{(\propto)}{}_{\omega}\varphi(s)g(s)\right|$$

$$= \left|\frac{1}{\Gamma(1+\alpha)}\int_{\vartheta}^{\omega}\left[\frac{1}{\Gamma(1+\alpha)}\int_x^t g(s)(ds)^{\propto}\right]\varphi^{(\propto)}(t)(dt)^{\propto}\right| \quad (7)$$

By using the generalized Holders disparity and the comprehensive convexity of $\left|\varphi^{(\propto)}\right|^{p/(p-1)}$, we get

$$\left|\varphi(\vartheta)_{\vartheta}I^{(\propto)}{}_{x}g(s)+\varphi(b)_{x}I^{(\propto)}{}_{\omega}g(s) - \vartheta I^{(\propto)}{}_{\omega}\varphi(s)g(s)\right|$$

$$\leq \frac{1}{\Gamma(1+\alpha)}\left[\frac{1}{\Gamma(1+\alpha)}\int_{\vartheta}^{\omega}\left|\frac{1}{\Gamma(1+\alpha)}\int_x^t g(s)(ds)^{\propto}\right|^{p}(dt)^{\propto}\right]^{\frac{1}{p}}\left[\int_{\vartheta}^{b}|(\varphi(t))^{\propto}|^{\frac{p}{p-1}}(dt)^{\propto}\right]^{\frac{p}{p-1}} = (M_1+M_2)^{\frac{1}{q}} \quad (8)$$

Now by using (3), we get

$$M_1 = \frac{1}{\Gamma(1+\alpha)}\left[\frac{1}{\Gamma(1+\alpha)}\int_{\vartheta}^{\omega}\left|\frac{1}{\Gamma(1+\alpha)}\int_x^t g(s)(ds)^{\propto}\right|^{p}(dt)^{\propto}\right]^{\frac{1}{p}}$$

$$\leq \|g(s)\|_{p,\propto}\frac{((x-t)^{\alpha})^{\frac{1}{p}}}{\Gamma(1+\alpha)} \quad (9)$$

also,

$$M_2 = \left[\int_{\vartheta}^{\omega}|(\varphi(t))^{\propto}|^{\frac{p}{p-1}}(dt)^{\propto}\right]^{\frac{p}{p-1}}$$

$$\leq \left(\frac{1}{\Gamma(1+\alpha)}\right)^{\frac{p}{p-1}}\left[\frac{1}{\Gamma(1+\alpha)}\int_{\vartheta}^{\omega}\left|(\varphi(t))^{\frac{\propto p-\propto}{p}}\right|(dt)^{\propto}\right]^{\frac{p}{p-1}}$$

By using lemma 2.1, we get

"$$\left[\int_{\vartheta}^{\omega}\left|(\varphi(t))^{\propto}\right|^{\frac{p}{p-1}}(dt)^{\propto}\right]^{\frac{p}{p-1}} \leq \left(\frac{1}{\Gamma(1+\alpha)}\right)^{\frac{p}{p-1}}\frac{\Gamma\left(1+\frac{p\propto-\propto}{p}\right)}{\Gamma\left(1+\frac{2\alpha p-\propto}{p}\right)}\left(\omega^{\frac{2\propto p-\propto}{p}} - \vartheta^{\frac{2\propto p-\propto}{p}}\right)" \quad (10)$$

Put (9) and (10) in (8), we get

$$\left|\varphi(\vartheta)_{a}I^{(\propto)}{}_{x}g(s)+\varphi(\omega)_{x}I^{(\propto)}{}_{b}g(s) - \vartheta I^{(\propto)}{}_{\omega}\varphi(s)g(s)\right| \leq \frac{((x-t)^{\alpha})^{\frac{1}{p}}}{\Gamma(1+\alpha)^{\frac{p^2+p-1}{p-1}}} \cdot \frac{\Gamma\left(\frac{p-\propto+\alpha p}{p}\right)}{\Gamma\left(\frac{p-\propto+2\alpha p}{p}\right)}\left(\omega^{\frac{2\propto p-\propto}{p}} - \vartheta^{\frac{2\propto p-\propto}{p}}\right)\|g(s)\|_{p,\propto}$$

$$where \quad 1 \leq p,q \leq \infty, \frac{1}{p}+\frac{1}{q}=1.$$

Corollary3.4:

If $g: [\vartheta, \omega] \to R^{\alpha}$ be a symmetric to $(\vartheta+\omega)/_2$. Then we have another new Hadamard inequality.

Then

$$\left|\frac{\varphi(\vartheta)+\varphi(\omega)}{2^{\propto}} \quad {}_{\vartheta}I^{(\propto)}{}_{\omega}g(s) + -\vartheta I^{(\propto)}{}_{\omega}\varphi(s)g(s)\right| \le \frac{\left(\left(\frac{\vartheta+\omega}{2}-t\right)^{\alpha}\right)^{\frac{1}{p}}}{\varphi\Gamma(1+\alpha)^{\frac{p^2+p-1}{p-1}}} \cdot \frac{\Gamma\left(\frac{p-\propto+\alpha p}{p}\right)}{\Gamma\left(\frac{p-\propto+2\alpha p}{p}\right)}\left(\omega^{\frac{2\propto p-\propto}{p}} - \vartheta^{\frac{2\propto p-\propto}{p}}\right)\|g(s)\|_{p,\propto}$$

$$where \qquad 1 \le p,q \le \infty, \frac{1}{p}+\frac{1}{q}=1.$$

Proof:

Put $x=\frac{\vartheta+b}{2}$ in theorem 3.3 , we come to be

$$\left|\varphi(\vartheta)_{a}I^{(\propto)}{}_{\frac{\vartheta+\omega}{2}}g(s)+\varphi(b)_{\frac{\vartheta+\omega}{2}}I^{(\propto)}{}_{\omega}g(s) - \vartheta I^{(\propto)}{}_{\omega}\varphi(s)g(s)\right| \le \frac{((x-t)^{\alpha})^{\frac{1}{p}}}{\Gamma(1+\alpha)^{\frac{p^2+p-1}{p-1}}} \cdot \frac{\Gamma\left(\frac{p-\propto+\alpha p}{p}\right)}{\Gamma\left(\frac{p-\propto+2\alpha p}{p}\right)}\left(\omega^{\frac{2\propto p-\propto}{p}} - \vartheta^{\frac{2\propto p-\propto}{p}}\right)\|g(s)\|_{p,\propto} \qquad (11)$$

Now,

Utilizing the regularity of g , the researchers will have the resulting characteristics.

$$\varphi(\vartheta)_{\vartheta}I^{(\propto)}{}_{\frac{\vartheta+\omega}{2}}g(s)+\varphi(\omega)_{\frac{\vartheta+\omega}{2}}I^{(\propto)}{}_{b}g(s) - \vartheta I^{(\propto)}{}_{\omega}\varphi(s)g(s) = \frac{\varphi(\vartheta)+\varphi(\omega)}{2^{\propto}} \quad {}_{\vartheta}I^{(\propto)}{}_{\omega}g(s) + -\vartheta I^{(\propto)}{}_{\omega}\varphi(s)g(s) \qquad (12)$$

Put (12) in (11), we change it to,

$$\left|\frac{\varphi(\vartheta)+\varphi(\omega)}{2^{\propto}} \quad {}_{\vartheta}I^{(\propto)}{}_{\omega}g(s) + -\vartheta I^{(\propto)}{}_{\omega}\varphi(s)g(s)\right| \le \frac{\left(\left(\frac{\vartheta+\omega}{2}-t\right)^{\alpha}\right)^{\frac{1}{p}}}{\Gamma(1+\alpha)^{\frac{P^2+P-1}{P-1}}} \cdot \frac{\Gamma\left(\frac{p-\propto+\alpha p}{p}\right)}{\Gamma\left(\frac{p-\propto+2\alpha p}{P}\right)}\left(b^{\frac{2\propto p-\propto}{p}} - \vartheta^{\frac{2\propto p-\propto}{p}}\right)\|g(s)\|_{p,\propto}.$$

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